HEAT EQUATIONS AND THE WEIGHTED $\bar\partial$-PROBLEM

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Abstract. The purpose of this article is to establish regularity and pointwise upper bounds for the (relative) fundamental solution of the heat equation associated to the weighted $\bar\partial$-operator in $L^2(\mathbb{C}^n)$ for a certain class of weights. The weights depend on a parameter, and we find pointwise bounds for heat kernel, as well as its derivatives in time, space, and the parameter. We also prove cancellation conditions for the heat semigroup. We reduce the n-dimensional case to the one-dimensional case, and the estimates in one-dimensional case are achieved by Duhamel's principle and commutator properties of the operators. As an application, we recover estimates of heat kernels on polynomial models in $\mathbb{C}^2$.

1. Introduction

The purpose of this article is to establish regularity and pointwise upper bounds for the (relative) fundamental solution of the heat equation associated to the weighted $\bar\partial$-operator in $L^2(\mathbb{C}^n)$ for a certain class of weights. The weights depend on a parameter, and we find bounds for heat kernel, as well as its derivatives in time, space, and the parameter. We also prove cancellation conditions for the heat semigroup. We reduce the n-dimensional case to the one-dimensional case, and the estimates in one-dimensional case will be achieved by a novel use of Duhamel's principle.

As an application of our estimates, we can recover and improve the estimates by Nagel and Stein in [16] for heat kernels on polynomial models in $\mathbb{C}^2$.

An additional point of interest is that the infinitesimal generator of the semigroup is a magnetic Schrödinger operator. Also, when the parameter is negative, this Schrödinger operator has a nonpositive and possibly unbounded electric potential, yet the large-time behavior of the semigroup is well-controlled (the eigenvalues of the generator are always nonnegative).

1.1. The set-up in one dimension – heat equations in $(0,\infty)\times\mathbb{C}$.

Let $p:\mathbb{C}\to\mathbb{R}$ be a subharmonic, nonharmonic polynomial and $\tau\in\mathbb{R}$ a parameter. Define

$$\overline{Z}_{\tau p,z} = \frac{\partial}{\partial\bar z} + \tau\frac{\partial p}{\partial \bar z},$$

a one-parameter family of differential operators acting on functions defined on $\mathbb{C}$. To solve the Cauchy-Riemann equations

$$\bar\partial u = f$$

for a function $f\in L^2(\mathbb{C},e^{-2\tau p}) = \{\varphi : \int_{\mathbb{C}}|\varphi|^2 e^{-2\tau p}\,dV<\infty\}$, it is equivalent to solve the weighted $\bar\partial$-problem

$$\overline{Z}_{\tau p,z}\alpha = \beta.$$

Our interest is studying the $\overline{Z}_{\tau p,z}$-problem through its associated heat equation (defined below). We wish to express the solution as an integral operator and finding the regularity and smoothness of the (relative) fundamental solution.

To study the $\overline{Z}_{\tau p}$-equation, we let

$$Z_{\tau p,z} = -\overline{Z}^*_{\tau p,z} = e^{\tau p}\frac{\partial}{\partial\bar z}e^{-\tau p} = \frac{\partial}{\partial z} - \tau\frac{\partial p}{\partial z}$$

introduce the $\overline{Z}_{\tau p}$-Laplacian

$$\Box_{\tau p,z} = -\overline{Z}_{\tau p,z}Z_{\tau p,z}.$$

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As observed in [21], if $\tilde{p}(x_1, x_2) = p(-x_1, -x_2)$ and $\widetilde{\Box}_{\tau p,z} = -Z_{\tau p,z}\overline{Z}_{\tau p,z}$, then

$$\widetilde{\Box}_{\tau\tilde{p},z} = \Box_{(-\tau)p,z}.$$

Thus, we will (mostly) restrict ourselves to the case $\tau > 0$ and study the heat equations

$$\begin{cases} \dfrac{\partial u}{\partial s} + \Box_{\tau p,z} u = 0 \\ u(0,z) = f(z) \end{cases} \tag{1}$$

and

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial s} + \widetilde{\Box}_{\tau p,z} \tilde{u} = 0 \\ \tilde{u}(0,z) = \tilde{f}(z). \end{cases} \tag{2}$$

We write $\Box_{\tau p}$ in lieu of $\Box_{\tau p,z}$ when the application is clear. From [19, 21], $\Box_{\tau p}$ and $\widetilde{\Box}_{\tau p}$ are self-adjoint, so from the spectral theorem and the Riesz Representation Theorem, we can express our solutions

$$u(s,z) = e^{-s\Box_{\tau p}}[f](z) = \int_{\mathbb{C}} H_{\tau p}(s,z,w) f(w)\, dA(w) \tag{3}$$

and

$$\tilde{u}(s,z) = e^{-s\widetilde{\Box}_{\tau p}}[f](z) = \int_{\mathbb{C}} \tilde{H}_{\tau p}(s,z,w) f(w)\, dA(w), \tag{4}$$

where $H_{\tau p}(s,z,w)$ and $\tilde{H}_{\tau p}(s,z,w)$ are C^∞ away from $\{(s,z,w) : s = 0 \text{ and } z = w\}$ (assuming $s \geq 0$, of course) and dA is Lebesgue measure on $\mathbb{C}$.

2. Discussion of the $n = 1$ case

We have several motivations for studying the heat equations (1) and (2).

2.1. $e^{-s\Box_b}$ on polynomial models in $\mathbb{C}^2$. A polynomial model M_p in $\mathbb{C}^2$ is a CR-manifold of the form

$$M_p = \{(z,w) \in \mathbb{C}^2 : \operatorname{Im} w = p(z)\}$$

where p is subharmonic, nonharmonic polynomial. M_p is the boundary of an unbounded pseudoconvex domain. When $p(z) = |z|^2$, M_p is the Heisenberg group $\mathbb{H}^1$, thus we can consider the Heisenberg group as the simplest example of a polynomial model.

The analysis of polynomial models in $\mathbb{C}^2$ is directly related to the $\overline{Z}_{\tau p}$-problem. $M_p \cong \mathbb{C} \times \mathbb{R}$, and if $w = t + ip(z)$, then $\bar{\partial}_b$ on M can be identified with the vector field $\bar{L} = \frac{\partial}{\partial \bar{z}} - i\frac{\partial p}{\partial \bar{z}}\frac{\partial}{\partial t}$. $\bar{L}$ is translation invariant in t, so if we take the partial Fourier transform in t where τ is the transform variable of t, then $\bar{L} \xmapsto{\widehat{\ }} \overline{Z}_{\tau p}$. Consequently, $\Box_{\tau p}$ and $\widetilde{\Box}_{\tau p}$ are the partial Fourier transforms of $\Box_b$ on (0,1)-forms and functions, respectively.

In $\mathbb{C}^2$, the $\Box_b$-heat equation on the polynomial model M_p was solved by Nagel and Stein [16]. They prove that the heat kernel of $e^{-s\Box_b}$ satisfies rapid decay. In Section 5, we improve their heat kernel estimates to Gaussian estimates in $|z - w|$. Street [23] has shown Gaussian decay with respect to the control metric for the $\Box_b$-heat kernel. His method does not seem to generalize to the $n \geq 2$ case, however, while ours ought to, as noted in Section 3.

In light of the Nagel-Stein result and the connection of $H_{\tau p}(s,z,w)$ and $\tilde{H}_{\tau p}(s,z,w)$ with the $\Box_b$-heat kernel on polynomial models on $\mathbb{C}^2$, it follows that $H_{\tau p}(s,z,w)$ and $H_{\tau p}(s,z,w)$ are actually C^∞ off the diagonal $\{(s,z,w,\tau) : s = 0,\ z = w\}$. Thus, the content of this article is to prove the decay estimates and cancellation conditions.

2.2. $\Box_{\tau p}$ **and** $\widetilde{\Box}_{\tau p}$ **as magnetic Schrödinger operators.** If $a = \tau(-\frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_1})$ and $V = \frac{\tau}{2}\triangle p$, then

$$2\Box_{\tau p} = \frac{1}{2}(i\nabla - a)^2 + V, \qquad 2\widetilde{\Box}_{\tau p} = \frac{1}{2}(i\nabla - a)^2 - V$$

magnetic Schrödinger operators with magnetic potential a and electric potential $\pm V$.

The operators $\Box_{\tau p}$ and $\widetilde{\Box}_{\tau p}$ behave quite differently. As discussed in [21], $\widetilde{\Box}_{\tau p}$ has a nonpositive and unbounded potential (unbounded if $\deg \triangle p \geq 1$). A further complication is that for $\tau > 0$, $\operatorname{null}(\widetilde{\Box}_{\tau p}) \neq \{0\}$ (and in fact may be infinite dimensional, see [6]) while $\operatorname{null}(\Box_{\tau p}) = \{0\}$. Fortunately, $\widetilde{\Box}_{\tau p}$ has nonnegative eigenvalues and is self-adjoint, so it follows from the spectral theorem that $\lim_{s\to\infty} e^{-s\widetilde{\Box}_{\tau p}} = S_{\tau p}$ where $S_{\tau p}$ is the Szegö projection, i.e., the projection of $L^2(\mathbb{C})$ onto $\operatorname{null}\overline{Z}_{\tau p}$. The Szegö projection is given by

$$S_{\tau p}[f](z) = \int_{\mathbb{C}} S_{\tau p}(z,w) f(w)\, dA(w)$$

and $S_{\tau p}(z,w) \in C^\infty(\mathbb{C})$ [6].

A consequence of the nonzero limit is that the kernel of $e^{-s\widetilde{\Box}_{\tau p}}$ cannot vanish as $s \to \infty$. Thus, $\int_0^\infty e^{-s\widetilde{\Box}_{\tau p}}\, ds$ diverges and cannot be the relative fundamental solution of $\widetilde{\Box}_{\tau p}$. $e^{-s\widetilde{\Box}_{\tau p}}(I - S_{\tau p})$ functions as the natural replacement for $e^{-s\widetilde{\Box}_{\tau p}}$ since $\int_0^\infty e^{-s\widetilde{\Box}_{\tau p}}(I - S_{\tau p})\, ds$ converges and equals the relative fundamental solution of $\widetilde{\Box}_{\tau p}$.

2.3. $\overline{Z}_{\tau p}$**,** $\Box_{\tau p}$**, and the weighted** $\bar\partial$**-operator in** $\mathbb{C}$**.** In [6], Christ studies the $\bar\partial$-problem in $L^2(\mathbb{C}, e^{-2p})$ by the $\overline{Z}_{\tau p}$-problem when $\tau = 1$. Christ solves the $\Box_p$-equation using methods different from ours. He shows that $\Box_p$ is invertible and that the solution can be written as integration against a fractional integral operator. He finds pointwise upper bounds on the integral kernel and related objects. For more background on the weighted $\bar\partial$-problem in $\mathbb{C}$, see [21, 6, 1, 8]

2.4. $\Box_{\tau p}$ **and Hartogs Domains in** $\mathbb{C}^2$**.** Mathematicians have analyzed operators on Hartogs domains in $\mathbb{C}^n$ by understanding weighted operators on their base spaces. The original operators are then reconstructed using Fourier series [12, 8, 2]. Recently, on a class of Hartogs domains $\Omega \subset \mathbb{C}^2$, Fu and Straube [9, 10] establish an equivalence between the compactness of the $\bar\partial$-Neumann problem and the blowup of the smallest eigenvalue of $\Box_{\tau p}$ as $\tau \to \infty$. Christ and Fu [7] build on the work of Fu and Straube to show that the following are equivalent: compactness of the $\bar\partial$-Neumann operator, compactness of the complex Green operator, and $b\Omega$ satisfying property (P).

3. The $n \geq 2$ case and its reduction to $n = 1$

3.1. **The** $\Box_{\bar D}$ **and** $\Box_{\bar D}$**-heat equations.** In $\mathbb{C}^n$, $n \geq 2$, the Cauchy-Riemann equations in weighted spaces take the form

$$\bar\partial u = f$$

where f is a $(0, q+1)$-form in $L^2(\mathbb{C}^n, e^{-2\lambda}) = \{\varphi : \int_{\mathbb{C}^n} |\varphi|^2 e^{-2\lambda}\, dV < \infty\}$. We can solve the weighted $\bar\partial$-equation by solving the equivalent unweighted problem

$$\bar D\alpha = \beta$$

where $\bar D = e^{-\lambda}\bar\partial e^{\lambda}$. Solving the $\bar\partial$-equation by solving a related weighted problem is a classical technique that goes back to Hörmander [11]. Our interest is not simply in solving the $\bar D$-problem, but expressing the solution as an integral operator and finding the regularity and smoothness of the (relative) fundamental solution. We work with the class of weights $\lambda = \tau P(z_1, \dots, z_n) = \tau \sum_{j=1}^n p_j(z_j)$ where $\tau \in \mathbb{R}$ is a parameter and p_j are subharmonic, nonharmonic polynomials. We call such polynomials P decoupled. For the remainder of the section, we assume that $\bar D$ is of the form

$$\bar D_{\tau P} = \bar D = e^{-\tau P}\bar\partial e^{\tau P}$$

where P is a decoupled polynomial and each p_k is subharmonic and nonharmonic.

To study the $\bar D$-equation, we let $\bar D^*$ be the L^2-adjoint of $\bar D$ and introduce the $\bar D$-Laplacian

$$\Box^q_{\bar D} = \Box_{\bar D} = \bar D^*\bar D + \bar D\bar D^*$$

on $(0,q)$-forms. We will study the $\Box_{\bar{D}}$-heat equation

$$\begin{cases} \frac{\partial u}{\partial s} + \Box_{\bar{D}} u = 0 \\ u(0,z) = f(z) \end{cases} \tag{5}$$

because (as discussed below) we can recover both the solution to $\Box_{\bar{D}}$-equation, $\Box_{\bar{D}}\alpha = \beta$, and the projection onto the null-space of $\Box_{\bar{D}}$. We call the projection the Szegö projection and denote it $S_{\Box_{\bar{D}}}$. Our goal is to find the (relative) fundamental solution to the $\Box_{\bar{D}}$-heat equation and express the solution

$$u(s,z) = \int_{\mathbb{C}^n} H_{\tau P}(s,z,w) f(w)\, dV(w)$$

where $H_{\tau P}(s,z,w)$ is the relative fundamental solution, hereafter called the $\Box_{\bar{D}}$-heat kernel. We wish to find the regularity and pointwise upper bounds of the $\Box_{\bar{D}}$-heat kernel and its derivatives in time (s), space (z and w), and the parameter (τ).

3.2. **$\Box_b$ on decoupled polynomial models in $\mathbb{C}^n$.** In [17], Nagel and Stein find optimal estimates for solutions to the Kohn Laplacian $\Box_b$ on a class of models in $\mathbb{C}^n$. They study hypersurfaces of the form $M_P = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} = P(z_1, \cdots, z_n)\}$ where $P(z_1, \dots, z_n) = p_1(z_1) + \cdots + p_{n-1}(z_{n-1})$ and p_j are subharmonic, nonharmonic polynomials. The surface M_P is the boundary of an unbounded pseudoconvex domain and is called a decoupled polynomial model. For example, if $p(z) = |z_1|^2 + \cdots + |z_n|^2$, then M_p is the Heisenberg group $\mathbb{H}^n$ and is the boundary of the Siegel upper half space, and in [3], we use Hermite functions to explicitly compute the Fourier transform of the $\Box_b$-heat kernel as well as the $\Box_{\bar{D}}$-heat kernel. In [4], Boggess and Raich present a calculation of the Fourier transform of the fundamental solution of the $\Box_b$-heat equation on quadric submanifolds $M \subset \mathbb{C}^n \times \mathbb{C}^m$. In particular, we compute the analog of the $\Box_{\bar{D}}$-heat kernel.

In analog to polynomial models in $\mathbb{C}^2$, the $M_P \cong \mathbb{C}^n \times \mathbb{R}$ since points in M_P are of the form $(z_1, \dots, z_n, t + iP(z))$. The realization of $\bar{\partial}_b$ (defined on M_p) on $\mathbb{C}^n \times \mathbb{R}$ is a translation invariant operator in t. The partial Fourier transform in t of the Kohn Laplacian $\Box_b$ (identified with its image on $\mathbb{C}^n \times \mathbb{R}$) is the $\bar{D}$-Laplacian. Thus, the $\Box_{\bar{D}}$-heat kernel is the partial Fourier transform of the $\Box_b$-kernel.

We would like to study the heat semigroup $e^{-s\Box_b}$ on M_P. A motivation for studying the heat kernel is that one of the most important aspects of [17] is that their qualitatively sharp estimates for $\Box_b$ are written in terms of both the control metric and the Szegö pseudometric (see [18, 5, 14] for background on these metrics). We would like to understand the appearance of both metrics in the estimates.

3.3. **Reduction from the case $n \geq 2$ to $n = 1$.** The reason that the $n \geq 2$ and $n = 1$ cases can be studied together is that the decoupling of the polynomial P allows the $\Box_{\bar{D}}$-heat kernel to be expressed as a product of the $\Box_{\tau p}$- and $\widetilde{\Box}_{\tau p}$-heat kernels. Consequently, the n-dimensional problem reduces to a one-dimensional problem. This method of expressing an n-dimensional heat kernels as a product of one-dimensional heat kernels was used by Boggess and Raich to present a simplified calculation of the heat kernel on the Heisenberg group [3].

To see the factoring, we need to compute $\Box_{\bar{D}}$. Let ϑ_q be the set of increasing q-tuples $I = (i_1, \dots, i_q)$ with $1 \leq i_1 < \cdots < i_q \leq n$. For $J \in \vartheta_q$, let

$$\Box_k^{J(k)} = \begin{cases} \Box_{\tau p_k} & k \in J \\ \widetilde{\Box}_{\tau p_k} & k \notin J \end{cases},$$

and

$$\Box_J = \sum_{k=1}^n \Box_k^{J(k)}.$$

If $f = \sum_{J \in \vartheta_q} f_J \, d\bar{z}_J$, then since $Z_{\tau p_j}$ commutes with $\bar{Z}_{\tau p_k}$ (here $p_j = p_j(z_j)$ and $p_k = p_k(z_k)$) if $j \neq k$, a standard computation shows

$$\Box_{\bar{D}} f = \sum_{J \in \vartheta_q} \Box_J f_J \, d\bar{z}_J.$$

Since $\square_{\bar{D}}$ is self-adjoint, we can solve the $\square_{\bar{D}}$-heat equation via the spectral theorem, i.e., $u(s,z) = e^{-s\square_{\bar{D}}}[f](z)$. Since $\square_{\bar{D}}$ acts diagonally, it is enough to study $\square_J$ acting on functions. $\square_J$ is a sum of commuting operators $\square_k^{J(k)}$, so

$$e^{-s\square_J} = e^{-s\sum_{k=1}^n \square_k^{J(k)}} = \prod_{k=1}^n e^{-s\square_k^{J(k)}}.$$

Each $e^{-s\square_k^{J(k)}}$ acts only in the z_k variable, and consequence of this fact is that the integral kernel of the product has a simpler form than (the analog of) a convolution. If $H_J(s,z,w)$ is the heat-kernel to the $\square_J$-heat equation, then

$$H_J(s,z,w) = \prod_{k=1}^n H_k^{J(k)}(s,z_k,w_k)$$

where

$$H_k^{J(k)}(s,z_k,w_k) = \begin{cases} H_{\tau p_k}(s,z_k,w_k) & \text{if } \square_k^{J(k)} = \square_{\tau p_k} \\ \tilde{H}_{\tau p_k}(s,z_k,w_k) & \text{if } \square_k^{J(k)} = \widetilde{\square}_{\tau p_k} \end{cases}.$$

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4. Results

Since the n-dimensional heat kernel can be written in terms of one-dimensional heat kernels, for simplicity we will write all of our results in terms of the one-dimensional kernels.

In [19, 21], we establish pointwise upper bounds for $H_{\tau p}(s,z,w)$ and $\tilde{H}_{\tau p}(s,z,w)$ and their space and time derivatives. The spectral theorem techniques in our earlier work are poorly suited for differentiating in the parameter, so we develop a new integral formula based on Duhamel's principle to handle derivatives in the parameter. The results proven in this article greatly extend our earlier results.

In order to write the estimates for $H_{\tau p}(s,z,w)$ and $\tilde{H}_{\tau p}(s,z,w)$, we need the appropriate differential operators. Since $\square_{\tau p}$ is a self-adjoint operator in $L^2(\mathbb{C})$, it follows that $H_{\tau p}(s,z,w) = \overline{H_{\tau p}(s,w,z)}$ [19]. Thus, in w, the appropriate differential operators are:

$$\overline{W}_{\tau p,w} = \overline{(Z_{\tau p,w})} = \frac{\partial}{\partial \bar{w}} - \tau\frac{\partial p}{\partial \bar{w}} = e^{\tau p}\frac{\partial p}{\partial \bar{w}}e^{-\tau p}, \qquad W_{\tau p,w} = \overline{(\overline{Z}_{\tau p,w})} = \frac{\partial}{\partial w} + \tau\frac{\partial p}{\partial w} = e^{-\tau p}\frac{\partial p}{\partial w}e^{\tau p}.$$

To motivate the correct differential operator in τ, it is essential to have the "twist" term

$$T(w,z) = -2\,\mathrm{Im}\Big(\sum_{j\geq 1}\frac{1}{j!}\frac{\partial^j p(z)}{\partial z^j}(w-z)^j\Big)$$

from the control metric on M_p. It turns out that the distance on M_p in the t-component is written in terms of $t+T(w,z)$, and the partial Fourier transform in t of $t+T(w,z)$ is the twisted derivative

$$M_{\tau p}^{z,w} = e^{i\tau T(w,z)}\frac{\partial}{\partial\tau}e^{-i\tau T(w,z)} = \frac{\partial}{\partial\tau} - iT(w,z).$$

Also associated to the control metric is the pseudo-distance

$$\mu_p(z,\delta) = \inf_{j,k\geq 1}\left|\frac{\delta}{\frac{1}{j!k!}\frac{\partial^{j+k}p(z)}{\partial z^j\partial\bar{z}^k}}\right|^{1/(j+k)}$$

and its approximate inverse

$$\Lambda(z,\delta) = \sum_{j,k\geq 1}\left|\frac{1}{j!k!}\frac{\partial^{j+k}p(z)}{\partial z^j\partial\bar{z}^k}\right||\delta|^{j+k}.$$

Roughly speaking, the volume of a ball in M_p of radius δ is approximately $\delta^2\Lambda(z,\delta)$ and the distance from a point (z,t) to (w,s) is $|z-w| + \mu_p(z,t-s+T(w,z))$. The functions μ and Λ satisfy

$$\mu_p\big(z,\Lambda(z,\delta)\big) \sim \Lambda\big(z,\mu_p(z,\delta)\big) \sim \delta.$$

Finally, for $(s,z) \in (0,\infty) \times \mathbb{C}$, define

$$\Delta = \min\{\mu_p(z, 1/\tau), s^{1/2}\}.$$

It will be important to distinguish the number of space derivatives from the number of τ-derivatives, and we do this by introducing the (n,ℓ)-differentiation classes for functions of $(\tau, z, w) \in \mathbb{R} \times \mathbb{C} \times \mathbb{C}$.

Definition 4.1. We say that Y^J is an (n,ℓ)-*derivative* and write $Y^J \in (n,\ell)$ if $Y^J = Y_{|J|}Y_{|J|-1}\cdots Y_1$ is a product of $|J|$ operators of the form $Y_j = \overline{Z}_{\tau p,z}, Z_{\tau p,z}, \overline{W}_{\tau p,w}, W_{\tau p,w}$, or $M^{z,w}_{\tau p}$ where $|J| = n+\ell$, $n = \#\{Y_j : Y_j = M^{z,w}_{\tau p}\}$ and $\ell = \#\{Y_j : Y_j = \overline{Z}_{\tau p,z}, Z_{\tau p,z}, \overline{W}_{\tau p,w}, W_{\tau p,w}\}$. Also, we write $Y^J \le (n,\ell)$ if $Y^J \in (k,j)$ where $k \le n$ and $j \le \ell$.

For $Y^J \in (n,\ell)$, if $0 \le \alpha \le |J|$, let $Y^{J-\alpha} = Y_{|J|-\alpha}Y_{|J|-1-\alpha}\cdots Y_1$. While this is an abuse of notation, we will only use it in situations where the length of the derivative is important. Also, we commonly use the notation X^J for operators $X^J \in (0,\ell)$ and Y^J when $Y^J \in (n,\ell)$, $n \ge 0$.

The main results for the heat equations of $\Box_{\tau p}$ and $\widetilde{\Box}_{\tau p}$ are the following.

Theorem 4.2. *Let p be a subharmonic, nonharmonic polynomial and $\tau > 0$ a parameter. If $Y^J \in (n,\ell)$ and $k \ge 0$, then there exist constants $C_{k,|J|}, c > 0$ so that*

$$\left|\frac{\partial^k}{\partial s^k} Y^J H_{\tau p}(s,z,w)\right| \le C_{k,|J|}\frac{\Lambda(z,\Delta)^n}{s^{1+k+\frac{1}{2}\ell}} e^{-c\frac{|z-w|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}}.$$

Since $\Lambda(z,\mu_p(z,1/\tau)) \sim 1/\tau$, we have the immediate corollary.

Corollary 4.3. *Let $\tau > 0$. If $Y^J \in (n,\ell)$ and $k \ge 0$, then there exist constants $C_{k,|J|}, c > 0$ so that*

$$\left|\frac{\partial^k}{\partial s^k} Y^J H_{\tau p}(s,z,w)\right| \le C_{k,|J|}\frac{1}{\tau^n s^{1+k+\frac{1}{2}\ell}} e^{-c\frac{|z-w|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}}.$$

Theorem 4.4. *Let p be a subharmonic, nonharmonic polynomial and $\tau > 0$ a parameter. If $Y^\alpha \in (n,\ell)$ and $k \ge 0$, then there exist positive constants $c, C_{|\alpha|}, C_{k,|\alpha|}$, so that*

$$\left|Y^\alpha \tilde{H}_{\tau p}(s,z,w)\right| \le C_{|\alpha|}\frac{\Lambda(z,\Delta)^n}{\Delta^{2+\ell}} e^{-c\frac{|z-w|^2}{s}} e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}} e^{-c\frac{|z-w|}{\mu_p(w,1/\tau)}}$$

Also, if the derivatives annihilate the Szegö kernel, i.e., $\frac{\partial^k}{\partial s^k} Y^\alpha S_{\tau p}(z,w) = 0$, then the estimate simplifies to

$$\left|\frac{\partial^k}{\partial s^k} Y^\alpha \tilde{H}_{\tau p}(s,z,w)\right| \le \frac{C_{k,|\alpha|}\Lambda(z,\Delta)^n}{s^{1+k+\frac{1}{2}\ell}} e^{-c\frac{|z-w|^2}{s}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}}.$$

Given the importance of Szegö projection to the relative fundamental solution of $\widetilde{\Box}_{\tau p}$, we will also study the integral kernel of $e^{-s\widetilde{\Box}_{\tau p}}(I - S_{\tau p})$ and its derivatives. Specifically, we write

$$e^{-s\widetilde{\Box}_{\tau p}}(I - S_{\tau p})[f](z) = \int_{\mathbb{C}} \tilde{G}_{\tau p}(s,z,w) f(w)\, dA(w)$$

and will analyze $\tilde{G}_{\tau p}(s,z,w)$ and its derivatives.

Theorem 4.5. *Let p be a subharmonic, nonharmonic polynomial and $\tau > 0$ a parameter. If $Y^\alpha \in (n,\ell)$ and $k \ge 0$, then there exist positive constants $c, C_{|\alpha|}, C_{k,|\alpha|}$ so that*

$$\left|Y^\alpha \tilde{G}_{\tau p}(s,z,w)\right| \le \frac{C_{|\alpha|}}{\tau^n} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}} e^{-c\frac{|z-w|}{\mu_p(w,1/\tau)}} \max\left\{\frac{e^{-c\frac{|z-w|^2}{s}}}{s^{1+\frac{1}{2}\ell}}, \frac{1}{\mu_p(w,1/\tau)^{2+\ell}}\right\}.$$

Also, when the derivatives annihilate the Szegö kernel, i.e., $\frac{\partial^k}{\partial s^k} Y^\alpha S_{\tau p}(z,w) = 0$, $\frac{\partial^k}{\partial s^k} Y^\alpha \tilde{H}_{\tau p}(s,z,w) = \frac{\partial^k}{\partial s^k} Y^\alpha \tilde{G}_{\tau p}(s,z,w)$ and the estimate is

$$\left|\frac{\partial^k}{\partial s^k} Y^\alpha \tilde{G}_{\tau p}(s,z,w)\right| \le C_{k,|\alpha|}\frac{\Lambda(z,\Delta)^n}{s^{1+k+\frac{1}{2}\ell}} e^{-c\frac{|z-w|^2}{s}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}}.$$

Remark 4.6. The $(0,\ell)$-case of Theorem 4.2, Theorem 4.4, and Theorem 4.5 is proved in [19, 21]. The fact that $\min_{s\geq 0} \frac{|z-w|^2}{s} + \frac{s}{\mu_p(w,1/\tau)^2} = \frac{|z-w|}{\mu_p(w,1/\tau)}$ which allows for the $\exp(-c\frac{|z-w|}{\mu_p(w,1/\tau)})\exp(-c\frac{|z-w|}{\mu_p(z,1/\tau)})$ term to be factored outside of the max.

The estimates in Theorem 4.4 and Theorem 4.5 generalize the main results in [21] in which we prove the estimates for the $(0,\ell)$-case.

As essential tool in the proof of Theorem 4.5 is a special case of Corollary 4.2 from [21]. The corollary in [21] is based on the identity $e^{-s\Box_{\tau p}}\overline{Z}_{\tau p} = \overline{Z}_{\tau p}e^{-s\tilde{\Box}_{\tau p}}$. If $R_{\tau p}$ is the relative inverse to $\overline{Z}_{\tau p}$ (i.e., $R_{\tau p}\overline{Z}_{\tau p} = I - S_{\tau p}$) and has integral kernel $R_{\tau p}(z,w)$, then $R_{\tau p}(z,w)$ is the relative fundamental solution to $\overline{Z}_{\tau p}$, and it is shown that applying $R_{\tau p}$ to the identity yields the following result.

Proposition 4.7. *Let $\tau > 0$. Then*

$$\overline{Z}_{\tau p,z}\tilde{H}_{\tau p}(s,z,w) = \overline{Z}_{\tau p,z}\tilde{G}_{\tau p}(s,z,w) = \overline{W}_{\tau p,w}H_{\tau p}(s,z,w),$$

and

$$W_{\tau p,w}\tilde{H}_{\tau p}(s,z,w) = W_{\tau p,w}\tilde{G}_{\tau p}(s,z,w) = Z_{\tau p,z}H_{\tau p}(s,z,w).$$

Also,

$$\tilde{G}_{\tau p}(s,z,w) = -\int_{\mathbb{C}} \overline{W}_{\tau p,w}H_{\tau p}(s,v,w)R_{\tau p}(z,v)\,dA(v).$$

From Theorem 4.2 and the first two equalities in Corollary 4.7, the latter statements in Theorem 4.4 and Theorem 4.5 follow immediately.

The estimates in Theorem 4.2 and Theorem 4.5 allow us to recover pointwise estimates on the Szegö kernel and the relative fundamental solution to $\overline{Z}_{\tau p}$. We need to integrate out s in the estimate of $Z_{\tau p,z}H_{\tau p}(s,z,w)$ to recover estimates for the relative fundamental solution of $\overline{Z}_{\tau p}$, and we need to take the limit as $s \to 0$ of the estimate for $\tilde{G}_{\tau p}$ to recover the estimate of the Szegö kernel. We have the corollary

Corollary 4.8. *Let $\tau > 0$. If Y^J is an (n,ℓ)-derivative, then there exist constants $C_{|J|}, c > 0$ so that*

$$|Y^J R_{\tau p}(z,w)| \leq C_{|J|}\begin{cases} \tau^{-n}|z-w|^{-\ell} & |z-w| \leq \mu_p(z,1/\tau) \\ \frac{1}{\tau^n \mu_p(z,1/\tau)^{1+|J|}} e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}} e^{-c\frac{|z-w|}{\mu_p(w,1/\tau)}} & |z-w| \geq \mu_p(z,1/\tau). \end{cases}$$

Also,

$$|Y^J S_{\tau p}(z,w)| \leq C_{|J|}\frac{1}{\tau^n \mu_p(z,1/\tau)^{2+\ell}} e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}} e^{-c\frac{|z-w|}{\mu_p(w,1/\tau)}}.$$

The proof of Corollary 4.8 is identical to the proof of Corollary 2 in [19].

Remark 4.9. The estimates in Theorem 4.4 and Theorem 4.5 are natural given the estimate in Theorem 4.2. Since $\lim_{s\to\infty} e^{-s\tilde{\Box}_{\tau p}} = S_{\tau p}$, the large time estimate in Theorem 4.4 should agree with the estimate for the Szegö kernel. From Corollary 4.8, we see that the estimates agree as $s \to \infty$. Similarly, since $\lim_{s\to 0} e^{-s\tilde{\Box}_{\tau p}}(I - S_{\tau p}) = I - S_{\tau p}$, the estimates for $\tilde{G}_{\tau p}(s,z,w)$ must become the estimates for $S_{\tau p}(z,w)$ as $s \to 0$. Thus, expressing the estimates in Theorem 4.4 and Theorem 4.5 in terms of maximums is natural – the estimates agree with the estimates for $\Box_{\tau p}$-heat kernel and the Szegö kernel on the appropriate regions.

We prove a cancellation condition for $e^{-s\Box_{\tau p}}$. It compliments the pointwise estimates of Theorem 4.2 and is of interest in its own right. Following the notation of [19, 21, 16], we let

$$H^s_{\tau p}[\varphi](z) = e^{-s\Box_{\tau p}}[\varphi](z)$$

and similarly for $\tilde{H}^s_{\tau p}[\varphi](z)$ and $\tilde{G}^s_{\tau p}[\varphi](z)$.

Theorem 4.10. *Let $\tau > 0$. If $Y^J \in (n,\ell)$, $\delta < \max\{\mu_p(z,1/\tau), s^{\frac{1}{2}}\}$, and $\varphi \in C^\infty_c\big(D(z,\delta)\big)$, then there exists a constant $C_{|J|}$ so that for ℓ even,*

$$|Y^J H^s_{\tau p}[\varphi](z)| \leq C_{|J|}\frac{\Lambda(z,\Delta)^n}{\delta}\big(\|\Box^{\frac{\ell}{2}}_{\tau p}\varphi\|_{L^2(\mathbb{C})} + \delta^2\|\Box^{\frac{\ell}{2}+1}_{\tau p}\varphi\|_{L^2(\mathbb{C})}\big),$$

and for ℓ odd,

$$|Y^J H^s_{\tau p}[\varphi](z)| \le C_{|J|}\frac{\Lambda(z,\Delta)^n}{\delta}\big(\delta\|\Box_{\tau p}^{\frac{\ell+1}{2}}\varphi\|_{L^2(\mathbb{C})} + \delta^3\|\Box_{\tau p}^{\frac{\ell+3}{2}}\varphi\|_{L^2(\mathbb{C})}\big),$$

Theorem 4.10 allows us to recover a cancellation condition for $G_{\tau p} = \Box_{\tau p}^{-1}$ and the relative fundamental solution of $\overline{Z}_{\tau p}$, denoted $R_{\tau p}$.

Corollary 4.11. *Let $\tau > 0$. Let $\delta > 0$ and $\varphi \in \mathcal{C}^\infty_c(D(z,\delta))$. Let $Y^\alpha \in (n,\ell)$. There exists a constant $C_{n,|\alpha|}$ so that if $|\alpha| = 2k > 0$ is even or $|\alpha| = 0$ and $\delta \ge \mu_p(z,1/\tau)$, then*

$$|Y^\alpha G_{\tau p}[\varphi](z)| \le \frac{C_{|\alpha|}}{\tau^n}\delta\big(\|\Box^k_{\tau p}\varphi\|_{L^2(\mathbb{C})} + \delta^2\|\Box^{k+1}_{\tau p}\varphi\|_{L^2(\mathbb{C})}\big)$$

and if $|\alpha| = 2k+1 > 0$ is odd, then

$$|Y^\alpha G_{\tau p}[\varphi](z)| \le \frac{C_{|\alpha|}}{\tau^n}\delta\big(\delta\|\Box^{k+1}_{\tau p}\varphi\|_{L^2(\mathbb{C})} + \delta^3\|\Box^{k+2}_{\tau p}\varphi\|_{L^2(\mathbb{C})}\big).$$

If $|\alpha| = 0$ and $\delta < \mu_p(z,1/\tau)$, then

$$|G_{\tau p}[\varphi](z)| \le \frac{C_0}{\tau^n}\delta\Big(\log(\tfrac{2\mu_p(z,1/\tau)}{\delta})\|\varphi\|_{L^2(\mathbb{C})} + \delta^2\|\Box_{\tau p}\varphi\|_{L^2(\mathbb{C})}\Big).$$

The proof of Corollary 4.11 can be followed line by line from the proof of Lemma 3.6 in [21].

5. Estimates for heat kernels for polynomial models in $\mathbb{C}^2$.

As discussed in §2.1, we have the polynomial model M_p. Let $H(s,p,q)$ be the integral kernel of $e^{-s\Box_b}$. Because M is a polynomial model, if $p = (z,t_1)$, $q = (w,t_2)$ and $t = t_1 - t_2$, then we can consider $H(s,p,q) = H(s,z,w,t)$ and $G(s,p,q) = G(s,z,w,t)$. In this notation, if $d_M(z,w,t) = |z-w| + \mu_p(z,t+T(w,z))$ and X^α and X^β are products of the vectors fields L and $\bar{L}$, then Nagel and Stein [16] prove that for any nonnegative integer N, there exists a constant $C_{N,\alpha,\beta,j}$ so that

$$\left|\frac{\partial^j}{\partial s^j}X^\alpha_p X^\beta_q H(s,z,w,t_1-t_2)\right| \le C_{N,\alpha,\beta,j}\frac{d_M(z,w,t)^{-2j-|\alpha|-|\beta|}}{d_M(z,w,t)^2\Lambda(z,d_M(z,w,t))}\left[\frac{s^N}{s^N + d_M(z,w,t)^{2N}}\right].$$

As a consequence of Theorem 4.2 and Theorem 4.4, we improve the previous estimates to the following.

Theorem 5.1. *Let $p : \mathbb{C} \to \mathbb{R}$ be a subharmonic, nonharmonic polynomial and $M_p = \{(z,w) \in \mathbb{C}^2 : \operatorname{Im} w = p(z)\}$. Under the standard identification of $M_p \cong \mathbb{C}\times\mathbb{R}$, if $H(s,z,w,t_1-t_2)$ is the integral kernel of $e^{-s\Box_b}$ and X^α and X^β are compositions of the vector fields L and $\bar{L}$, then for any integer N, there exists a constants $c = c_{N,\alpha,\beta,j}$ and $C_{N,\alpha,\beta,j}$ so that*

$$\left|\frac{\partial^j}{\partial s^j}X^\alpha_p X^\beta_q H(s,z,w,t_1-t_2)\right| \le C_{N,\alpha,\beta,j}\frac{e^{-c\frac{|z-w|^2}{s}}}{d_M(z,w,t)^{2+2j+|\alpha|+|\beta|}\Lambda(z,d_M(z,w,t))}\frac{s^N}{\mu_p(z,t+T(w,z))^{2N}}$$

where $t = t_1 - t_2$.

Proof. We sketch the proof in the $\alpha = \beta = j = 0$ case. The other cases follow similarly. By a partial Fourier transform,

$$H(s,z,w,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-it\tau}H_{\tau p}(s,z,w)\,d\tau.$$

Next,

$$\left|\int_{\mathbb{R}} e^{-it\tau}H_{\tau p}(s,z,w)\,d\tau\right| = \frac{1}{|t+T(w,z)|^n}\left|\int_{\mathbb{R}} e^{-it\tau}M^n_{\tau p}H_{\tau p}(s,z,w)\,d\tau\right|.$$

Note that $\mu_p(z,1/\tau) \ge s^{1/2}$ is equivalent to $\tau \le \Lambda(z,s^{1/2})^{-1}$. By Theorem 4.2 and Theorem 4.4, we estimate

$$\begin{aligned}&\frac{1}{|t+T(w,z)|^n}\left|\int_{|\tau|\le\Lambda(z,s^{1/2})^{-1}} e^{-it\tau}M^n_{\tau p}H_{\tau p}(s,z,w)\,d\tau\right|\\ &\qquad\le \frac{1}{|t+T(w,z)|^n}\int_{|\tau|\le\Lambda(z,s^{1/2})^{-1}}\frac{\Lambda(z,s^{1/2})^n}{s}e^{-c\frac{|z-w|^2}{2}}\,d\tau \le \frac{\Lambda(z,s^{1/2})^n}{|t+T(z,w)|^n}\frac{e^{-c\frac{|z-w|^2}{s}}}{s\Lambda(z,s^{1/2})}.\end{aligned}$$

Similarly, with n large enough so that the integral converges, by Theorem 4.4

$$\frac{1}{|t+T(w,z)|^n}\left|\int_{|\tau|\geq\Lambda(z,s^{1/2})^{-1}} e^{-it\tau} M_{\tau p}^n H_{\tau p}(s,z,w)\,d\tau\right|$$
$$\frac{1}{|t+T(w,z)|^n}\int_{|\tau|\geq\Lambda(z,s^{1/2})^{-1}} \frac{e^{-c\frac{|z-w|^2}{s}}}{|\tau|^n\mu_p(z,1/\tau)^2}\,d\tau \leq C\frac{\Lambda(z,s^{1/2})^n}{|t+T(z,w)|^n}\frac{e^{-c\frac{|z-w|^2}{s}}}{s\Lambda(z,s^{1/2})}.$$

There are two key ideas to finish the proof. The first is that the bound $\frac{\Lambda(z,s^{1/2})^n}{|t+T(z,w)|^n}$ for all n is equivalent to having the bound $\frac{s^N}{\mu_p(z,t+T(w,z))^{2N}}$ for all N (it is a matter is expanding the $\mu_p(z,t+T(w,z))$ terms and reshuffling the $\frac{\partial^{j+k}p(z)}{\partial z^j\partial\bar z^k}$ terms. The second key fact is that the statement n cannot equal zero is a manifestation of the lack of decay in s for $H(s,z,w,t)$. It turns out that to achieve the term $\frac{s^N}{\mu_p(z,t+T(w,z))^{2N}}$ in the estimate for $H(s,z,w,t)$ causes the replacement of s with $d(z,w,t)^2$ throughout the denominator. $\square$

If S is the Szegö projection of $L^2(M)$ on $\ker(\bar\partial_b)$, we can also recover estimates for the kernel of $e^{-s\Box_b}(I-S)$ and the Szegö projection. The difference in the argument is that we only integrate by parts only when τ is away from zero. The estimates themselves are less subtle as they do not involve rapid or exponential decay. The estimates on the Szegö projection are computed in both [15] and [13].

6. The nonhomogenous IVP, uniqueness, and mixed derivatives of the heat kernel

For the remainder of the article, we assume that $\tau>0$.

The key to expressing derivatives of $H_{\tau p}(s,z,w)$ in terms of quantities we can estimate is to solve the nonhomogeneous heat equation via Duhamel's principle.

6.1. Uniqueness of solutions of the nonhomogeneous IVP.

Proposition 6.1. *Let $\tau\in\mathbb{R}$. Let $g:(0,\infty)\times\mathbb{C}\to\mathbb{C}$ and $f:\mathbb{C}\to\mathbb{C}$ be $H^2(\mathbb{C})$ for each s and vanish as $|z|\to\infty$. The solution to the nonhomogeneous heat equation*

$$\begin{cases}\dfrac{\partial u}{\partial s}+\Box_{\tau p}u=g \text{ in } (0,\infty)\times\mathbb{C}\\ \lim\limits_{s\to0}u(s,z)=f(z)\end{cases}\tag{6}$$

is given by

$$u(s,z)=\int_{\mathbb{C}}H_{\tau p}(s,z,\xi)f(\xi)\,dA(\xi)+\int_0^s\int_{\mathbb{C}}H_{\tau p}(s-r,z,\xi)g(r,\xi)\,dA(\xi)dr.\tag{7}$$

Proof. By [19], it suffices to show that $u(s,z)=\int_0^s\int_{\mathbb{C}}H_{\tau p}(s-r,z,\xi)g(r,\xi)\,dA(\xi)dr$ solves (6) when $f\equiv0$. Let

$$u_\epsilon(s,z)=\int_0^s\int_{\mathbb{C}}H_{\tau p}(s-r+\epsilon,z,\xi)g(\xi,r)\,dA(\xi)dr.$$

Then

$$\frac{\partial u_\epsilon}{\partial s}=\int_0^s\int_{\mathbb{C}}\frac{\partial H_{\tau p}}{\partial s}(s-r+\epsilon,z,\xi)g(r,\xi)\,dA(\xi)dr+\int_{\mathbb{C}}H_{\tau p}(\epsilon,z,\xi)g(s,\xi)\,dA(\xi)$$

and

$$\Box_{\tau p,z}u_\epsilon(s,z)=\int_0^s\int_{\mathbb{C}}\Box_{\tau p,z}H_{\tau p}(s-r+\epsilon,z,\xi)g(r,\xi)\,dA(\xi)dr=\int_0^s\int_{\mathbb{C}}-\frac{\partial H_{\tau p}}{\partial s}(s-r+\epsilon,z,\xi)g(r,\xi)\,dA(\xi)dr.$$

Adding the previous two equations together, we have (let $g_s(z)=g(s,z)$)

$$\left(\frac{\partial u_\epsilon}{\partial s}+\Box_{\tau p,z}u_\epsilon\right)(s,z)=e^{-\epsilon\Box_{\tau p}}[g_s](z)\xrightarrow{\epsilon\to0}g(s,z)$$

in $L^2(\mathbb{C})$ for each fixed s. If $g_s\in C^2_c(\mathbb{C})$, then the convergence is uniform (as a consequence of [19]). We need to show that $\lim_{\epsilon\to0}u_\epsilon(s,z)=u(s,z)$. But this follows from writing

$$u_\epsilon(s,z)-u(s,z)=\epsilon\int_0^s\int_{\mathbb{C}}g(r,\xi)\frac{H_{\tau p}(s-r+\epsilon,z,\xi)-H_{\tau p}(s-r,z,\xi)}{\epsilon}\,dA(\xi)dr$$

and using the size and cancellation conditions for $\frac{\partial H_{\tau p}}{\partial s}$ in Theorem 4.2 Theorem 4.10 (the $(0,\ell)$-case is proved in [21]). The final fact we must check is that $\lim_{s\to 0} u(s,z) = 0$ in $L^2(\mathbb{C})$. This, however, follows from the fact that $e^{-s\Box_{\tau p}}$ is a contraction in L^2 for all $s \geq 0$. $\Box$

We next prove a uniqueness result for solutions of (6).

Proposition 6.2. *Let $\tau > 0$ and u_1 and u_2 satisfy (6). If $u_1, u_2 \in C^1\big((0,\infty)\times\mathbb{C}\big)$ and are in $L^2(\mathbb{C})$ for each s, then $u_1 = u_2$.*

Proof. We will use the fact from [6] that for $\tau > 0$, there exists $C = C(\tau, p) > 0$ so that $\|f\|_{L^2(\mathbb{C})} \leq C\|Z_{\tau p} f\|_{L^2(\mathbb{C})}$. Since u_1, u_2 satisfy (6), it follows that $h(s,z) = u_1(s,z) - u_2(s,z)$ satisfies $(\frac{\partial}{\partial s} + \Box_{\tau p})h = 0$ and $h(0,z) = 0$. Let $g(s) = \int_{\mathbb{C}} |h(s,z)|^2\, dA(z)$. Note that $g(s) \geq 0$ and $\frac{\partial h}{\partial s} = \overline{Z}_{\tau p} Z_{\tau p} h$. Consequently,

$$\begin{aligned} g'(s) &= 2\,\mathrm{Re}\Big(\int_{\mathbb{C}} \frac{\partial h}{\partial s}(s,z)\overline{h(s,z)}\, dA(z)\Big) \\ &= 2\,\mathrm{Re}\Big(\int_{\mathbb{C}} \overline{Z}_{\tau p} Z_{\tau p} h(s,z)\overline{h(s,z)}\, dA(z)\Big) = -2\int_{\mathbb{C}} |Z_{\tau p} h(s,z)|^2\, dA(z) \leq 0. \end{aligned}$$

Since g is nonnegative, $g(0) = 0$ and $g'(s) \leq 0$ for all s, it follows that $g(s) = 0$ for all s. Thus, $h(s,z) = 0$ for all $s, z \in \mathbb{C}$. $\Box$

6.2. **Mixed Derivatives of $H_{\tau p}(s,z,w)$.** We now derive a formula to express $Y^J H_{\tau p}(s,z,w)$ using Duhamel's principle.

Proposition 6.3. *Let $\tau \in \mathbb{R}$ and $Y^J \in (n,\ell)$. If*

$$H^J_{\tau p}(s,\xi,w) = \sum_{k=0}^{|J|-2} \Big(\prod_{\imath=0}^{k} Y_{|J|-\imath}\Big)\big[\Box_{\tau p,\xi}, Y_{|J|-k-1}\big] Y^{J-k-2} H_{\tau p}(s,\xi,w) + \big[\Box_{\tau p,\xi}, Y_{|J|}\big] Y^{J-1} H_{\tau p}(s,\xi,w) \tag{8}$$

where $\prod_{\imath=0}^{k} Y_{|J|-\imath} = Y_{|J|}Y_{|J|-1}\cdots Y_{|J|-k}$, then

$$Y^J H_{\tau p}(s,z,w) = \lim_{\epsilon\to 0}\int_{\mathbb{C}} H_{\tau p}(s,z,\xi) Y^J H_{\tau p}(\epsilon,\xi,w)\, dA(\xi) + \int_0^s \int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi) H^J_{\tau p}(r,\xi,w)\, dA(\xi) dr. \tag{9}$$

Proof. Observe that $\frac{\partial}{\partial s}$ commutes with Y_j for all j, so

$$\begin{aligned} &\Big(\frac{\partial}{\partial s} + \Box_{\tau p,\xi}\Big) Y^J H_{\tau p}(s,\xi,w) = Y_{|J|}\Big(\frac{\partial}{\partial s} + \Box_{\tau p,\xi}\Big) Y^{J-1} H_{\tau p}(s,\xi,w) + \big[\Box_{\tau p,\xi}, Y_{|J|}\big] Y^{J-1} H_{\tau p}(s,\xi,w) \\ &= Y_{|J|}Y_{|J|-1}\Big(\frac{\partial}{\partial s} + \Box_{\tau p,\xi}\Big) Y^{J-2} H_{\tau p}(s,\xi,w) + Y_{|J|}\big[\Box_{\tau p,\xi}, Y_{|J|-1}\big] Y^{J-2} H_{\tau p}(s,\xi,w) \\ &+ \big[\Box_{\tau p,\xi}, Y_{|J|}\big] Y^{J-1} H_{\tau p}(s,\xi,w) \\ &= \cdots = Y^J\Big(\frac{\partial}{\partial s} + \Box_{\tau p,\xi}\Big) H_{\tau p}(s,\xi,w) + \sum_{k=0}^{|J|-2} \Big(\prod_{\imath=0}^{k} Y_{|J|-\imath}\Big)\big[\Box_{\tau p,\xi}, Y_{|J|-k-1}\big] Y^{J-k-2} H_{\tau p}(s,\xi,w) \\ &+ \big[\Box_{\tau p,\xi}, Y_{|J|}\big] Y^{J-1} H_{\tau p}(s,\xi,w). \end{aligned}$$

But $H_{\tau p}(s,\xi,w)$ is annihilated by the heat operator, so

$$\Big(\frac{\partial}{\partial s} + \Box_{\tau p,\xi}\Big) Y^J H_{\tau p}(s,\xi,w) = H^J_{\tau p}(s,\xi,w).$$

$Y^J H_{\tau p}(s+\epsilon, z, w)$ satisfies (6) with initial value $Y^J H_{\tau p}(\epsilon, z, w)$ and inhomogeneous term $H^J_{\tau p}(s,z,w)$. By Proposition 6.1, sending $\epsilon \to 0$ in the second integral finishes the proof of the result. $\Box$

In order to use Proposition 6.3, we must understand $\big[\Box_{\tau p,z}, Y_j\big]$. Certainly, if $Y_j = \overline{W}_{\tau p,w}$ or $W_{\tau p,w}$, $\big[\Box_{\tau p,z}, Y_j\big] = 0$. For the other cases, it is helpful to recall the expansions of $\Box_{\tau p}$ and $\widetilde{\Box}_{\tau p}$. Recall that

$$\begin{aligned} \Box_{\tau p} &= -\frac{\partial^2}{\partial z \partial \bar z} + \tau\frac{\partial^2 p}{\partial z \partial \bar z} + \tau^2 \frac{\partial p}{\partial z}\frac{\partial p}{\partial \bar z} + \tau\left(\frac{\partial p}{\partial z}\frac{\partial}{\partial \bar z} - \frac{\partial p}{\partial \bar z}\frac{\partial}{\partial z}\right) \\ &= -\frac{1}{4}\triangle + \frac{1}{4}\tau\triangle p + \frac{\tau^2}{4}|\nabla p|^2 + \frac{i}{2}\tau\left(\frac{\partial p}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial p}{\partial x_2}\frac{\partial}{\partial x_1}\right) \end{aligned} \tag{10}$$

and

$$\widetilde{\Box}_{\tau p} = \frac{\partial^2}{\partial z \partial \bar{z}} - \tau \frac{\partial^2 p}{\partial z \partial \bar{z}} + \tau^2 \frac{\partial p}{\partial z}\frac{\partial p}{\partial \bar{z}} + \tau\left(\frac{\partial p}{\partial z}\frac{\partial}{\partial \bar{z}} - \frac{\partial p}{\partial \bar{z}}\frac{\partial}{\partial z}\right) \tag{11}$$
$$= -\frac{1}{4}\triangle - \frac{1}{4}\tau\triangle p + \frac{\tau^2}{4}|\nabla p|^2 + \frac{i}{2}\tau\left(\frac{\partial p}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial p}{\partial x_2}\frac{\partial}{\partial x_1}\right).$$

As noted above, $\Box_{\tau p}$ and $\widetilde{\Box}_{\tau p}$ are self-adjoint operators in $L^2(\mathbb{C})$, and since they are not real, their adjoints in the sense of distributions do not agree with their L^2-adjoints (on any functions in both domains, e.g., functions in $\mathcal{C}_c^\infty(\mathbb{C})$). If X is a differential operator, denote its distributional adjoint by $X^\#$. Note that $\overline{Z}^\#_{\tau p} = -\overline{W}_{\tau p}$ and $Z^\#_{\tau p} = -W_{\tau p}$.

Proposition 6.4. *Let*

$$e(w,\xi) = \sum_{j\geq 1} \frac{1}{j!}\frac{\partial^{j+1}p(\xi)}{\partial \xi^j \partial \bar{\xi}}(w-\xi)^j.$$

Since $M_{\tau p} f(\xi, w) = e^{i\tau T(w,\xi)}\frac{\partial}{\partial \tau}e^{-i\tau T(w,\xi)} f(\xi,w)$, we have the following:

(a) $\big[\Box_{\tau p,\xi}, \overline{Z}_{\tau p,\xi}\big] = -2\tau \dfrac{\partial^3 p}{\partial \xi \partial \bar{\xi}^2} - 2\tau \dfrac{\partial^2 p}{\partial \xi \partial \bar{\xi}} \overline{Z}_{\tau p,\xi}$.

(b) $\big[\Box_{\tau p,\xi}, Z_{\tau p,\xi}\big] = 2\tau \dfrac{\partial^2 p}{\partial \xi \partial \bar{\xi}} Z_{\tau p,\xi}$.

(c) $\big[\Box_{\tau p,\xi}, M^{\xi,w}_{\tau p}\big] = -\dfrac{\partial^2 p}{\partial \xi \partial \bar{\xi}} - e(w,\xi) Z_{\tau p,\xi} + \overline{e(w,\xi)}\,\overline{Z}_{\tau p,\xi}$.

(d) $[M^{\xi,w}_{\tau p}, \overline{Z}_{\tau p,\xi}] = -e(w,\xi)$.

(e) $[M^{\xi,w}_{\tau p}, Z_{\tau p,\xi}] = \overline{e(w,\xi)}$.

Proof. The proof is a computation. Parts (d) and (e) are proven directly from the definitions. For (a),

$$\big[\Box_{\tau p,\xi}, \overline{Z}_{\tau p,\xi}\big] = Z_{\tau p,\xi}(\widetilde{\Box}_{\tau p,\xi} - \Box_{\tau p,\xi}) = \overline{Z}_{\tau p,\xi}\Big(-2\tau\frac{\partial^2 p}{\partial \xi \partial \bar{\xi}}\Big) = -2\tau \frac{\partial^3 p}{\partial \xi \partial \bar{\xi}^2} - 2\tau \frac{\partial^2 p}{\partial \xi \partial \bar{\xi}} \overline{Z}_{\tau p,\xi}.$$

(b) is a similar computation. To prove (c), observe that

$$[\Box_{\tau p,\xi}, M^{\xi,w}_{\tau p}]f = [M^{\xi,w}_{\tau p}, \overline{Z}_{\tau p,\xi}]Z_{\tau p,\xi}f - \overline{Z}_{\tau p,\xi}[Z_{\tau p,\xi}, M^{\xi,w}_{\tau p}]f = -e(w,\xi)Z_{\tau p}f + \overline{e(w,\xi)}\,\overline{Z}_{\tau p,\xi}f + \frac{\partial(\overline{e(w,\xi)})}{\partial \bar{\xi}}f,$$

and $\frac{\partial(\overline{e(w,\xi)})}{\partial \xi} = -\frac{\partial^2 p(\xi)}{\partial \xi \partial \xi}$, and the result follows from (d) and (e). $\Box$

Proposition 6.4 underscores the importance of conjugating the τ-derivative by an oscillating factor. $\frac{\partial p}{\partial \xi}$ and $\frac{\partial p}{\partial \xi}$ are not controlled by Λ_p and μ_p while $|e(w,\xi)(w-\xi)| \leq \Lambda(\xi, |w-\xi|)$.

6.3. **Digression – Discussion of Strategy.** We will prove Theorem 4.2 and Theorem 4.10 in part by a double induction. In order to understand the double induction, it is helpful to investigate (8) and (9) further. Let $Y^J \in (n,\ell)$. From [19], we have the desired bounds for $n=0$ and ℓ arbitrary. Thus, it is natural to induct on n.

Assume that $Y^J = X^\alpha (M^{z,w}_{\tau p})^n$. From (9), it is clear that we will have to understand

$$\big(\prod_{\imath=0}^{k} Y_{|J|-\imath}\big)[\Box_{\tau p,\xi}, Y_{|J|-k-1}]Y^{J-k-2}H_{\tau p}(s,\xi,w).$$

If $Y_{|J|-k-1} = \overline{W}_{\tau p,w}$ or $W_{\tau p,w}$, then the commutator is 0 because the terms commute. If $Y_{|J|-k-1} = M^{z,w}_{\tau p}$, then

$$\Big(\prod_{\imath=0}^{k} Y_{|J|-\imath}\Big)[\square_{\tau p,\xi}, M^{\xi,w}_{\tau p}]Y^{J-k-2}H_{\tau p}(s,\xi,w) = \Big(\prod_{\imath=0}^{k} Y_{|J|-\imath}\Big)\Big(-\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar{\xi}}\Big)Y^{J-k-2}H_{\tau p}(s,\xi,w)$$
$$+\Big(\prod_{\imath=0}^{k} Y_{|J|-\imath}\Big)\Big(e(w,\xi)Z_{\tau p,\xi} + \overline{e(w,\xi)}\overline{Z}_{\tau p,\xi}\Big)Y^{J-k-2}H_{\tau p}(s,\xi,w).$$

All three of the terms in the right-hand side involve $(n-1)$ appearances of $M^{\xi,w}_{\tau p}$, so presumably they can be controlled by the induction hypothesis on n.

If, however, $Y_{|J|-k-1} = Z_{\tau p}$, then $[\square_{\tau p}, Z_{\tau p}] = 2\tau\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar{\xi}}Z_{\tau p,\xi}$, then we must estimate

$$\int_0^s\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)\Big(\prod_{\imath=0}^{k} Y_{|J|-\imath}\Big)\tau\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar{\xi}}Z_{\tau p,\xi}Y^{|J|-k-2}H_{\tau p}(r,\xi,w)\,dA(\xi)dr.$$

As written, this integral is not good because the second term in the integral is not covered by the induction hypothesis – $M^{\xi,w}_{\tau p}$ appears n times. However, there exists a ξ-derivative term before any $M^{\xi,w}_{\tau p}$-term , so we can integrate by parts. Specifically, $(\prod_{\imath=0}^{k} Y_{|J|-\imath})\tau\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar{\xi}}Z_{\tau p,\xi}Y^{|J|-k-2}$ can be written in one of the following two forms: $X_w^{\alpha_1}X_\xi X^{\alpha_2}\big(\tau\frac{\partial^2 p}{\partial\xi\partial\bar{\xi}}\big)Z_{\tau p,\xi}Y^{|J|-k-2}$ or $X_w^{\alpha_1}\tau\frac{\partial^2 p}{\partial\xi\partial\bar{\xi}}Z_{\tau p,\xi}Y^{|J|-k-2}$. The form depends on whether or not $Z_{\tau p,\xi}$ is the first term that involves taking a ξ derivative. The integral then becomes

$$\begin{aligned}
&\int_0^s\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)\Big(\prod_{\imath=0}^{k} Y_{|J|-\imath}\Big)\tau\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar{\xi}}Z_{\tau p,\xi}Y^{|J|-k-2}H_{\tau p}(r,\xi,w)\,dA(\xi)dr\\
&= \int_0^s\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)X_w^{\alpha_1}X_\xi X^{\alpha_2}\big(\tau\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar{\xi}}\big)Z_{\tau p,\xi}Y^{|J|-k-2}H_{\tau p}(r,\xi,w)\,dA(\xi)dr\\
&= \int_0^s\int_{\mathbb{C}} X_\xi^{\#}H_{\tau p}(s-r,z,\xi)X_w^{\alpha_1}X^{\alpha_2}\tau\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar{\xi}}Z_{\tau p,\xi}Y^{|J|-k-2}H_{\tau p}(r,\xi,w)\,dA(\xi)dr
\end{aligned}$$

or

$$\begin{aligned}
&\int_0^s\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)\Big(\prod_{\imath=0}^{k} Y_{|J|-\imath}\Big)\tau\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar{\xi}}Z_{\tau p,\xi}Y^{|J|-k-2}H_{\tau p}(r,\xi,w)\,dA(\xi)dr\\
&= \int_0^s\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)X_w^{\alpha_1}\big(\tau\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar{\xi}}\big)Z_{\tau p,\xi}Y^{|J|-k-2}H_{\tau p}(r,\xi,w)\,dA(\xi)dr\\
&= -\int_0^s\int_{\mathbb{C}} W_{\tau p,\xi}\Big[\tau\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar{\xi}}H_{\tau p}(s-r,z,\xi)\Big]X_w^{\alpha_1}Y^{|J|-k-2}H_{\tau p}(r,\xi,w)\,dA(\xi)dr
\end{aligned}$$

In both integrals, the number of $M^{\xi,w}_{\tau p}$ terms remains n, but the number of derivatives in w and ξ is $(\ell-1)$. This suggests that we ought to induct in ℓ, so the $(n,\ell-1)$-case is covered by the induction hypothesis.

Our goal is pointwise estimates of $Y^J H_{\tau p}(s,z,w)$. The complicating factor is that the induction hypothesis needs to include both a size estimate *and* a cancellation condition.

6.4. **Preliminary Computations.** It is convenient to use the shorthand $a^z_{jk} = \frac{1}{j!k!}\frac{\partial^{j+k}p(z)}{\partial z^j\partial\bar{z}^k}$. With this notation,

$$p(w) = \sum_{j,k\geq 0} a^z_{jk}(w-z)^j\overline{(w-z)}^k.$$

The following estimates will be useful.

Lemma 6.5. *Let $c>0$ and $\epsilon>0$. With a decrease in c, we have the bounds*

(a) $e^{-c\frac{|\xi-w|^2}{s}}e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^\epsilon}|\nabla^\ell_{\xi,w}e(w,\xi)| \lesssim e^{-c\frac{|\xi-w|^2}{s}}e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^\epsilon}\tau^{-1}\min\{s^{-\frac{\ell}{2}-\frac{1}{2}},\mu_p(\xi,\frac{1}{\tau})^{-\ell-1}\}$.

(b) $e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^\epsilon}|\nabla^\ell_\xi\triangle p(\xi)| \lesssim e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^\epsilon}\tau^{-1}\min\{s^{-\frac{\ell}{2}-1},\mu_p(\xi,\frac{1}{\tau})^{-\ell-2}\}$.

(c)

$$e^{-c\frac{|\xi-w|^2}{s}} e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^\epsilon} e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^\epsilon} |\nabla_\xi^\ell \triangle p(\xi)| \lesssim e^{-c\frac{|\xi-w|^2}{s}} e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^\epsilon} e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^\epsilon} \tau^{-1} \min\{s^{-\frac{\ell}{2}-1}, \mu_p(w,\tfrac{1}{\tau})^{-\ell-2}\}$$

(d)

$$e^{-c\frac{|\xi-w|^2}{s}} e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^\epsilon} e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^\epsilon} |\nabla_{\xi,w}^\ell e(w,\xi)| \lesssim e^{-c\frac{|\xi-w|^2}{s}} e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^\epsilon} e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^\epsilon} \tau^{-1} \min\{s^{-\frac{\ell}{2}-\frac{1}{2}}, \mu_p(w,\tfrac{1}{\tau})^{-\ell-1}\}.$$

Proof. First,

$$|\nabla_{\xi,w}^\ell e(w,\xi)| \le \sup_{\substack{j+k-1-\ell\ge 0\\ j,k\ge 1}} |a_{jk}^\xi||\xi-w|^{j+k-1-\ell}. \tag{12}$$

and

$$|\nabla_\xi^\ell \triangle p(\xi)| \sim |a_{jk}^\xi|$$

for some j,k satisfying $\ell+2=j+k$ and $j,k\ge 1$.

Next, with a decrease in c,

$$e^{-c\frac{|\xi-w|^2}{s}} e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^\epsilon} \le C_{\alpha,\beta} \frac{s^\alpha}{|\xi-w|^{2\alpha}} \frac{\mu_p(\xi,1/\tau)^{2\beta}}{s^\beta} e^{-c\frac{|\xi-w|^2}{s}} e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^\epsilon}. \tag{13}$$

Since $\Lambda\big(\xi,\mu_p(\xi,1/\tau)\big)\sim 1/\tau$, combining (12) and (13) finishes the proof of (a). The proof of (b) is simpler. $|\tau a_{jk}^\xi| \le \mu_p(\xi,1/\tau)^{-j-k}$. For the other inequality, use the previous inequality and (13) with $\alpha=0$ and $\beta=\frac{1}{2}(j+k)$.

The proofs of (c) and (d) are similar. We can write $p(\xi)=\sum_{\alpha,\beta\ge 0} a_{\alpha\beta}^w(\xi-w)^\alpha\overline{(\xi-w)}^\beta$, so

$$|a_{jk}^\xi| = \Big|c_{j,k}\sum_{\substack{\alpha\ge j\\ \beta\ge k}} c_{\alpha,\beta,j,k} a_{\alpha\beta}^w (\xi-w)^{\alpha-j}\overline{(\xi-w)}^{\beta-k}\Big| \lesssim \sup_{\substack{\alpha\ge j\\ \beta\ge k}} |a_{\alpha\beta}^w|\xi-w|^{\alpha+\beta-j-k}. \tag{14}$$

Since $|\nabla^\ell\triangle p(\xi)|\sim|a_{jk}^\xi|$ for some j,k satisfying $j\ge 1$, $k\ge 1$, $j+k=\ell+2$, it follows that (with a decrease in c)

$$\begin{aligned}
&\Big|e^{-c\frac{|\xi-w|^2}{s}} e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^\epsilon} e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^\epsilon} a_{jk}^\xi\Big| \\
&\le e^{-c\frac{|\xi-w|^2}{s}} e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^\epsilon} e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^\epsilon} \sup_{\substack{\alpha\ge j\\ \beta\ge k}} |a_{\alpha\beta}^w||\xi-w|^{\alpha+\beta-j-k} \frac{s^{\frac{1}{2}(\alpha+\beta-j-k)}}{|\xi-w|^{\alpha+\beta-j-k}} \frac{\mu_p(w,1/\tau)^{\alpha+\beta-j-k}}{s^{\frac{1}{2}(\alpha+\beta-j-k)}} \\
&\le e^{-c\frac{|\xi-w|^2}{s}} e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^\epsilon} e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^\epsilon} \frac{1}{\mu_p(w,1/\tau)^{j+k}} \Lambda(w,\mu_p(w,\tfrac{1}{\tau})) \\
&\sim e^{-c\frac{|\xi-w|^2}{s}} e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^\epsilon} e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^\epsilon} \frac{1}{\tau\mu_p(w,1/\tau)^{\ell+2}}.
\end{aligned}$$

The other cases are handled similarly. □

Proposition 6.6.

$$e(w,\xi) = -e(\xi,w) - \sum_{j,k\ge 1}\frac{1}{j!k!}\frac{\partial^{j+k+1}p(w)}{\partial w^j\partial\bar w^{k+1}}(\xi-w)^j\overline{(\xi-w)}^k = -\sum_{\substack{j\ge 1\\ k\ge 0}}\frac{1}{j!k!}\frac{\partial^{j+k+1}p(w)}{\partial w^j\partial\bar w^{k+1}}(\xi-w)^j\overline{(\xi-w)}^k.$$

Proof. Since $p(\xi)=\sum_{m,n\ge 0}\frac{1}{m!n!}\frac{\partial^{m+n}p(w)}{\partial w^m\partial\bar w^n}(\xi-w)^m\overline{(\xi-w)}^n$, it follows that

$$\frac{\partial^{j+1}p(\xi)}{\partial\xi^j\partial\bar\xi} = \sum_{\substack{m\ge j\\ n\ge 1}}\frac{1}{(m-j)!(n-1)!}\frac{\partial^{m+n}p(w)}{\partial w^m\partial\bar w^n}(\xi-w)^{m-j}\overline{(\xi-w)}^{n-1}.$$

Thus,

$$e(w,\xi)=\sum_{m,n\geq 1}\Big(\sum_{j=1}^{m}\binom{m}{j}(-1)^j\Big)\frac{1}{m!(n-1)!}\frac{\partial^{m+n}p(w)}{\partial w^m\partial\bar w^n}(\xi-w)^m\overline{(\xi-w)}^{n-1},$$

the desired equality since $\sum_{j=1}^m\binom{m}{j}(-1)^j=-1$ if $m\geq 1$. $\square$

Another useful equality is

$$a^z_{jk}=\frac{1}{j!k!}\sum_{\substack{m\geq j\\ n\geq k}}a^w_{mn}(z-w)^{m-j}\overline{(z-w)}^{n-k}. \tag{15}$$

An immediate consequence of (15) is that $\Lambda(z,|w-z|)\leq e^2\Lambda(w,|z-w|)$ and

$$\Lambda(z,|w-z|)\sim\Lambda(w,|z-w|). \tag{16}$$

The e^2 appears because $\sum_{j,k\geq 1}\frac{1}{j!k!}\leq e^2$.

Corollary 6.7. *Let $s>0$ and $z,w\in\mathbb{C}$ so that $|z-w|\leq s^{1/2}$. There exists a constant C depending on $\deg p$ so that if $\xi\in\mathbb{C}$, the following holds with a decrease in c:*

(i) $\big|e^{-c\frac{|z-\xi|^2}{s}}\nabla^m e(w,\xi)\big|\leq Ce^{-c\frac{|z-\xi|^2}{s}}s^{-(m+1)/2}\Lambda(z,s^{1/2})$.

(ii) $\big|e^{-c\frac{|z-\xi|^2}{s}}\Lambda(\xi,s^{1/2})\big|\leq Ce^{-c\frac{|z-\xi|^2}{s}}\Lambda(z,s^{1/2})$.

Proof. Proof of (i). If $|\xi-z|>|\xi-w|$, then by (15)

$$\begin{aligned}\big|e^{-c\frac{|z-\xi|^2}{s}}\nabla^m e(w,\xi)\big|&\leq e^{-c\frac{|z-\xi|^2}{s}}\sum_{j\geq m+1}|a^\xi_{j1}||\xi-z|^{j-m}\\&\leq e^{-c\frac{|z-\xi|^2}{s}+1}\sum_{j+k\geq m+1}|a^z_{jk}||\xi-z|^{j+k-m-1}\lesssim e^{-c\frac{|z-\xi|^2}{s}}s^{-\frac12(m+1)}\Lambda(z,s^{\frac12}).\end{aligned}$$

with a decrease in the constant c. If $|\xi-w|>2|z-w|$, then $|\xi-w|\sim|\xi-z|$, and we can use the same argument just given. If $|\xi-z|<|\xi-w|<2|z-w|$, then $\frac12|z-w|<|\xi-w|<2|z-w|$. Therefore, by Proposition 6.6 and (15),

$$\big|\nabla^m e(w,\xi)\big|\lesssim\sum_{j+k\geq m+1}|a^w_{jk}||z-w|^{j+k-m-1}\sim\sum_{j+k\geq m+1}|a^z_{jk}||z-w|^{j+k-m-1}\leq s^{-\frac12(m+1)}\Lambda(z,s^{1/2}).$$

(ii) is proved using similar methods, namely with (15) and the exponential decay. $\square$

7. Cancellation Conditions and Size Estimates for $Y^Je^{-s\Box_{\tau p}}$

We start by defining objects which will allow us to control the numerology in the induction.

Definition 7.1. Let n and ℓ be nonnegative integers. We say that $H_{\tau p}(s,z,w)$ satisfies the (n,ℓ)-*size conditions* if there exist positive constants $C_{|J|}$, c, and $\epsilon=\epsilon(n,\ell)$ so that for any $Y^J\in(n,\ell)$,

$$|Y^JH_{\tau p}(s,z,w)|\leq C_{|J|}\frac{\Lambda(z,\Delta)^n}{s^{1+\frac12\ell}}e^{-c\frac{|z-w|^2}{s}}e^{-c\left(\frac{s}{\mu_p(z,1/\tau)^2}\right)^\epsilon}e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^\epsilon}. \tag{17}$$

We say that $H_{\tau p}(s,z,w)$ satisfies the (n,∞)-*size conditions* if $H_{\tau p}(s,z,w)$ satisfies the (n,ℓ)-size conditions for all integers $\ell\geq 0$.

Notice that the (n,ℓ)-size condition is not as good as the decay in Theorem 4.2. It is, however, what the induction argument yields.

Definition 7.2. Let n and ℓ be nonnegative integers and $Y^J\in(n,\ell)$. We say that $H^s_{\tau p}$ satisfies the (n,ℓ)-*cancellation conditions* if there exists a constant $C_{\ell,n}$ so that for any $(s,z)\in(0,\infty)\times\mathbb{C}$ with $\delta\leq\max\{\mu_p(z,1/\tau),s^{\frac12}\}$ and $\varphi\in\mathcal{C}^\infty_c(D(z,\delta))$, then

$$|Y^JH^s_{\tau p}[\varphi](z)|=\left|\int_{\mathbb{C}}Y^JH_{\tau p}(s,z,w)\varphi(w)\,dA(w)\right|\leq C_{|J|}\frac{\Lambda(z,\Delta)^n}{\delta}\left(\big\|\Box^{\frac{\ell}{2}}_{\tau p}\varphi\big\|_{L^2(\mathbb{C})}+\delta^2\big\|\Box^{\frac{\ell}{2}+1}_{\tau p}\varphi\big\|_{L^2(\mathbb{C})}\right) \tag{18}$$

if ℓ is even and

$$|Y^J H^s_{\tau p}[\varphi](z)| = \left| \int_{\mathbb{C}} Y^J H_{\tau p}(s,z,w)\varphi(w)\, dA(w) \right| \le C_{|J|} \frac{\Lambda(z,\Delta)^n}{\delta} \big(\delta \big\|\Box_{\tau p}^{\frac{\ell+1}{2}} \varphi\big\|_{L^2(\mathbb{C})} + \delta^3 \big\|\Box_{\tau p}^{\frac{\ell+1}{2}+1}\varphi\big\|_{L^2(\mathbb{C})}\big). \tag{19}$$

if ℓ is odd. We say that $H^s_{\tau p}$ satisfies the (n,∞)-*cancellation conditions* if $H^s_{\tau p}$ satisfies the (n,ℓ)-cancellation conditions for all $\ell \ge 0$.

If φ satisfies the hypotheses of the test function in Definition 7.2, we will call φ a cancellation test function.

7.1. **Reduction of the General Case.** As discussed above, we will prove the size and cancellation conditions by inducting on both n and ℓ. In this spirit, we reduce the problem from analyzing $Y^J H_{\tau p}(s,z,w)$ and $Y^J H^s_{\tau p}[\varphi]$ for a general $Y^J \in (n,\ell)$ to Y^J of the form

$$Y^J = X^{J_1}(M^{z,w}_{\tau p})^n. \tag{20}$$

Since Theorem 4.2 and Theorem 4.10 are already proved for $n=0$ [19, 21], we can assume that $n \ge 1$. Moreover, $Y^J = (M^{z,w}_{\tau p})^n$ is already in the desired form, so we may assume that $\ell \ge 1$ and Y^J is not in the form $X^{J_1}(M^{z,w}_{\tau p})^n$.

For $k < \ell$, assume that $H^s_{\tau p}$ and $H_{\tau p}(s,z,w)$ satisfy the $(n-1,k)$-cancellation and size conditions, respectively. Since $n,\ell \ge 1$, we can write $Y^J = Y^{I_1} M^{z,w}_{\tau p} X Y^{I_2}$. The technique for handling $X = Z_{\tau p,z}$ or $\overline{Z}_{\tau p,z}$ is the same, so we will assume that $X = Z_{\tau p,z}$. In this case, by Proposition 6.4,

$$Y^{I_1} M^{z,w}_{\tau p} Z_{\tau p,z} Y^{I_2} = Y^{I_1} Z_{\tau p,z} M^{z,w}_{\tau p} Y^{I_2} + Y^{I_1}\overline{e(w,z)}Y^{I_2}.$$

The second term can be written as a sum of $(n-1,k)$-derivatives where $k \le \ell-1$. Using Lemma 6.5, it is straightforward exercise to use the $(n-1,k)$-size and cancellation conditions to check that $Y^{I_1}\overline{e(w,z)}Y^{I_2}H^s_{\tau p}$ and $Y^{I_1}\overline{e(w,z)}Y^{I_2}H_{\tau p}(s,z,w)$ also satisfy the (n,ℓ)-cancellation and size conditions. We iterate the commutation process for $Y^{I_1} Z_{\tau p,z} M^{z,w}_{\tau p} Y^{I_2}$, dealing with all of the error terms from the commutators as before. Thus, we can reduce Y^J to the form in (20).

7.2. **Computation of $\int_{\mathbb{C}} H_{\tau p}(s,z,\xi) Y^J H_{\tau p}(\epsilon,\xi,w)\, dA(\xi)$.** If $Y^J = X^{J_1}(M^{z,w}_{\tau p})^n$, then $X^{J_1} = U^\alpha_w X^\beta_z$ where $U = -X^{\#}$ and $X^{J_1} \in (0,|J_1|)$. The reason that we reduce Y^J to $U^\alpha_w X^\beta_z (M^{z,w}_{\tau p})^n$ is the following theorem and corollary.

Theorem 7.3. *Let $\tau \in \mathbb{R}$, $n \ge 1$ and $Y^J = U^\alpha_w X^\beta_z (M^{z,w}_{\tau p})^n \in (n,\ell)$. Then*

$$\lim_{\epsilon\to 0} \int_{\mathbb{C}} H_{\tau p}(s,z,\xi) U^\alpha_w X^\beta_\xi (M^{\xi,w}_{\tau p})^n H_{\tau p}(\epsilon,\xi,w)\, dA(\xi) = (-1)^{|\beta|} U^\alpha_w \big[r(w,\xi,z)^n U^\beta_\xi H_{\tau p}(s,z,\xi)\big]\Big|_{\xi=w}$$

where $r(w,\xi,z) = 2\,\mathrm{Im}\,\big(\sum_{j,k\ge 1} a^\xi_{jk}(w-\xi)^j \overline{(z-\xi)}^k\big)$.

Using the $(0,\ell)$-bounds from Theorem 4.2 and Theorem 4.10 and Lemma 6.5, the following corollary shows that $\lim_{\epsilon\to 0}\int_{\mathbb{C}} H_{\tau p}(s,z,\xi) U^\alpha_w X^\beta_x (M^{\xi,w}_{\tau p})^n H_{\tau p}(\epsilon,\xi,w)\, dA(\xi)$ satisfies the bounds for the (n,ℓ)-size and cancellation conditions.

Corollary 7.4. *Let U^α_w, X^β_z, and n be as in Theorem 7.3. If*

$$f_{\tau p}(s,z,w) = (-1)^{|\beta|} U^\alpha_w \big[r(w,\xi,z)^n U^\beta_\xi H_{\tau p}(s,z,\xi)\big]\Big|_{\xi=w},$$

then:

1. *$|f_{\tau p}(s,z,w)|$ is bounded by the right hand side of (17).*
2. *If φ is a cancellation test function, then $\big|\int_{\mathbb{C}} f_{\tau p}(s,z,w)\varphi(w)\, dA(w)\big|$ is bounded by (18) or (19), depending on whether $|\alpha|+|\beta|$ is even or odd.*

The proof of Theorem 7.3 is essentially combinatorial, and we have to establish some facts first.

We have the following:

Proposition 7.5.

$$T(w,z) = T(w,\xi) + T(\xi,z) - r(w,\xi,z).$$

To prove Proposition 7.5, we need the following combinatorial fact.

Lemma 7.6. *Fix integers k and n so that $0 \leq k \leq n$. Then*

$$\sum_{j=k}^{n}(-1)^j\binom{n}{j}\binom{j}{k} = \begin{cases}(-1)^n & k=n\\ 0 & k<n\end{cases}$$

Proof. Let $s(n,k) = \sum_{j=k}^{n}(-1)^j\binom{n}{j}\binom{j}{k}$. The $k=0$ case is standard. Indeed for $n>0$, the $k=0$ case follows from the expansion of $(x+y)^n$ with $x=1$ and $y=-1$. Recall that $\binom{n+1}{j} = \binom{n}{j}+\binom{n}{j-1}$. If $k \geq 1$, then

$$\begin{aligned}s(n+1,k) &= \sum_{j=k}^{n+1}(-1)^j\binom{j}{k}\left[\binom{n}{j}+\binom{n}{j-1}\right] = \sum_{j=k}^{n}(-1)^j\binom{n}{j}\binom{j}{k} + \sum_{j=k}^{n+1}(-1)^j\binom{n}{j-1}\binom{j}{k}\\ &= s(n,k) + \sum_{j=k-1}^{n}(-1)^{j-1}\binom{n}{j}\left(\binom{j}{k}+\binom{j}{k-1}\right) = s(n,k) - \sum_{j=k-1}^{n}(-1)^j\binom{n}{j}\binom{j}{k} - s(n,k-1)\\ &= s(n,k) - s(n,k) - s(n,k-1).\end{aligned}$$

Thus, $s(n,k) = -s(n,k-1) = \cdots = (-1)^k s(n-k,0)$, and the results follows from the $k=0$ case. ☐

With our combinatorial lemma in hand, we prove Proposition 7.5.

Proof. (Proposition 7.5). Recall that $T(w,z) = -2\operatorname{Im}\left(\sum_{j\geq 1}\frac{1}{j!}\frac{\partial^j p(z)}{\partial z^j}(w-z)^j\right)$. The strategy is to expand $\frac{\partial^j p(z)}{\partial z^j}$ about ξ. $p(z) = \sum_{n,\ell\geq 0}\frac{1}{n!\ell!}\frac{\partial^{n+\ell}p(\xi)}{\partial\xi^n\partial\bar\xi^\ell}(z-\xi)^n\overline{(z-\xi)}^\ell$. Then

$$\frac{\partial^j p(z)}{\partial z^j} = \sum_{\substack{n\geq j\\ \ell\geq 0}}\frac{1}{(n-j)!\ell!}\frac{\partial^{n+\ell}p(\xi)}{\partial\xi^n\partial\bar\xi^\ell}(z-\xi)^{n-j}\overline{(z-\xi)}^\ell,$$

so by Lemma 7.6,

$$\begin{aligned}\sum_{j\geq 0}\frac{1}{j!}\frac{\partial^j p(z)}{\partial z^j}(w-z)^j &= \sum_{j\geq 0}\sum_{\substack{n\geq j\\ \ell\geq 0}}\frac{1}{(n-j)!\ell!}\frac{\partial^{n+\ell}p(\xi)}{\partial\xi^n\partial\bar\xi^\ell}(z-\xi)^{n-j}\overline{(z-\xi)}^\ell\big((w-\xi)+(\xi-z)\big)^j\\ &= \sum_{j\geq 0}\sum_{\substack{n\geq j\\ \ell\geq 0}}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}\frac{1}{n!\ell!}\frac{\partial^{n+\ell}p(\xi)}{\partial\xi^n\partial\bar\xi^\ell}(z-\xi)^{n-j}\overline{(z-\xi)}^\ell(w-\xi)^k(\xi-z)^{j-k}\\ &= \sum_{n,\ell\geq 0}\sum_{k=0}^{n}\left(\sum_{j=k}^{n}(-1)^j\binom{n}{j}\binom{j}{k}\right)\frac{(-1)^k}{n!\ell!}\frac{\partial^{n+\ell}p(\xi)}{\partial\xi^n\partial\bar\xi^\ell}(z-\xi)^{n-k}\overline{(z-\xi)}^\ell(w-\xi)^k\\ &= \sum_{n,\ell\geq 0}\frac{1}{n!\ell!}\frac{\partial^{n+\ell}p(\xi)}{\partial\xi^n\partial\bar\xi^\ell}\overline{(z-\xi)}^\ell(w-\xi)^n\\ &= \sum_{n\geq 0}\frac{1}{n!}\frac{\partial^n p(\xi)}{\partial\xi^n}(w-\xi)^n + \sum_{\ell\geq 0}\frac{1}{\ell!}\frac{\partial^\ell p(\xi)}{\partial\bar\xi^\ell}\overline{(z-\xi)}^\ell + \sum_{n,\ell\geq 1}\frac{1}{n!\ell!}\frac{\partial^{n+\ell}p(\xi)}{\partial\xi^n\partial\bar\xi^\ell}\overline{(z-\xi)}^\ell(w-\xi)^n\end{aligned}$$

From [20], $T(w,\xi) = -2\operatorname{Im}\left(\sum_{j\geq 0}\frac{1}{j!}\frac{\partial^j p(\xi)}{\partial\xi^j}(w-\xi)^j\right)$ and $T(z,\xi) = -T(\xi,z)$, and we have

$$\begin{aligned}T(w,z) &= T(w,\xi) - T(z,\xi) - 2\operatorname{Im}\Big(\sum_{j,k\geq 1}a^\xi_{jk}(w-\xi)^j\overline{(z-\xi)}^k\Big)\\ &= T(w,\xi) + T(\xi,z) - r(w,\xi,z).\end{aligned}$$

☐

The following combinatorial results will help us with the bookkeeping in the proof of Theorem 7.3

Proposition 7.7. *For $n \geq 0$, let $0 \leq k \leq n$ and $0 \leq j \leq n-k$. Let $\{\gamma_j^{n,k}\}$ be a set of numbers so that $\gamma_0^{0,0} = 1$ and $\gamma_{-1}^{n,k} = \gamma_j^{n,-1} = 0$ for all j,k,n. If $\gamma_j^{n,k}$ satisfy the recursion relation*

$$\gamma_j^{n,k} = \gamma_j^{n-1,k} - \gamma_{j-1}^{n-1,k} - \gamma_j^{n-1,k-1},$$

then

$$\gamma_j^{n,k} = (-1)^{j+k} \binom{n}{k} \binom{n-k}{j}.$$

Proof. We induct on $n \geq 1$. $\gamma_0^{1,0} = 1$, $\gamma_1^{1,0} = -1$, and $\gamma_0^{1,1} = -1$, as predicted. Assume the result holds at level $(n-1)$. Then

$$\begin{aligned}
\gamma_j^{n,k} &= (-1)^{j+k} \binom{n-1}{k} \binom{n-k-1}{j} - (-1)^{j+k+1} \binom{n-1}{k} \binom{n-k-1}{j-1} - (-1)^{j+k-1} \binom{n-1}{k-1} \binom{n-k}{j} \\
&= (-1)^{j+k} \frac{(n-1)!}{(k-1)!(n-j-k-1)!(j-1)!} \left(\frac{1}{jk} + \frac{1}{k(n-j-k)} + \frac{1}{j(n-k-j)} \right) \\
&= (-1)^{j+k} \binom{n}{k} \binom{n-k}{j}.
\end{aligned}$$

□

We are now ready to prove Theorem 7.3.

Proof. (Theorem 7.3). The plan is to strip away $M_{\tau p}$ terms from $H_{\tau p}(\epsilon, \xi, w)$. We cannot, however, integrate by parts since there is no τ-integral. We can, however, use the product rule and Proposition 7.5 to effectively transfer $M_{\tau p}$ away from $H_{\tau p}(\epsilon, \xi, w)$. For clarity, since $M_{\tau p}$ will be applied to three different terms, we will use $M_{\tau p}^{u,v}$ to denote $e^{i\tau T(v,u)} \frac{\partial}{\partial \tau} e^{-i\tau T(v,u)}$. The proof is based on a repeated application of the following process.

$$\begin{aligned}
f(z,\xi,\tau) M_{\tau p}^{\xi,w} H_{\tau p}(\epsilon,\xi,w) &= \frac{\partial}{\partial \tau} \Big(f(z,\xi,\tau) H_{\tau p}(\epsilon,\xi,w) \Big) - \frac{\partial}{\partial \tau} f(z,\xi,\tau)\, H_{\tau p}(\epsilon,\xi,w) \\
&\quad - iT(w,z) f(z,\xi,\tau) H_{\tau p}(\epsilon,\xi,w) + iT(\xi,z) f(z,\xi,\tau) H_{\tau p}(\epsilon,\xi,w) - r(w,\xi,z) f(z,\xi,\tau) H_{\tau p}(\epsilon,\xi,w) \\
&= M_{\tau p}^{z,w} \Big(f(z,\xi,\tau) H_{\tau p}(\epsilon,\xi,w) \Big) - M_{\tau p}^{z,\xi} f(z,\xi,\tau)\, H_{\tau p}(\epsilon,\xi,w) - r(w,\xi,z) f(z,\xi,\tau) H_{\tau p}(\epsilon,\xi,w). \quad (21)
\end{aligned}$$

We now integrate by parts and use (21) repeatedly.

$$\begin{aligned}
&\int_{\mathbb{C}} H_{\tau p}(s,z,\xi) U_w^\alpha X_\xi^\beta (M_{\tau p}^{\xi,w})^n H_{\tau p}(\epsilon,\xi,w)\, dA(\xi) = (-1)^{|\beta|} U_w^\alpha \gamma_0^{0,0} \int_{\mathbb{C}} U_\xi^\beta H_{\tau p}(s,z,\xi) (M_{\tau p}^{\xi,w})^n H_{\tau p}(\epsilon,\xi,w)\, dA(\xi) \\
&= (-1)^{|\beta|} U_w^\alpha \gamma_0^{0,0} \Bigg(M_{\tau p}^{z,w} \int_{\mathbb{C}} U_\xi^\beta H_{\tau p}(s,z,\xi) (M_{\tau p}^{\xi,w})^{n-1} H_{\tau p}(\epsilon,\xi,w)\, dA(\xi) \\
&+ \int_{\mathbb{C}} \Big(- M_{\tau p}^{z,\xi} U_\xi^\beta H_{\tau p}(s,z,\xi) - r(w,\xi,z) U_\xi^\beta H_{\tau p}(s,z,\xi) \Big) (M_{\tau p}^{\xi,w})^{n-1} H_{\tau p}(\epsilon,\xi,w)\, dA(\xi) \Bigg) \\
&= (-1)^{|\beta|} \sum_{k=0}^{1} \sum_{j=0}^{1-k} \gamma_j^{1,k} U_w^\alpha (M_{\tau p}^{z,w})^{1-j-k} \int_{\mathbb{C}} r(w,\xi,z)^k (M_{\tau p}^{z,\xi})^j U_\xi^\beta H_{\tau p}(s,z,\xi) (M_{\tau p}^{\xi,w})^{n-1} H_{\tau p}(\epsilon,\xi,w)\, dA(\xi) \\
&= \cdots = (-1)^{|\beta|} \sum_{k=0}^{n} \sum_{j=0}^{n-k} \gamma_j^{n,k} U_w^\alpha (M_{\tau p}^{z,w})^{n-j-k} \int_{\mathbb{C}} r(w,\xi,z)^k (M_{\tau p}^{z,\xi})^j U_\xi^\beta H_{\tau p}(s,z,\xi) H_{\tau p}(\epsilon,\xi,w)\, dA(\xi) \\
&= (-1)^{|\beta|} \sum_{k=0}^{n} \sum_{j=0}^{n-k} \gamma_j^{n,k} \overline{X_w^\alpha (M_{\tau p}^{w,z})^{n-j-k} e^{-\epsilon \Box_{\tau p}} \left[\overline{r(w,\cdot,z)}^k (M_{\tau p}^{\cdot,z})^j X_\xi^\beta H_{\tau p}(s,\cdot,z) \right](w)}. \quad (22)
\end{aligned}$$

The problem with (22) is that we cannot commute $M_{\tau p}$ across $e^{-\epsilon\Box_{\tau p}}$. However, we can control the error term caused by the commutation. For a function $f = f_\tau(z,w)$, we have

$$\begin{aligned}
&M^{z,w}_{\tau p} e^{-\epsilon\Box_{\tau p}}[f_\tau(\cdot,w)](z) \\
&= e^{-\epsilon\Box_{\tau p}}[M^{\cdot,w}_{\tau p} f_\tau(\cdot,w)](z) + \epsilon M^{z,w}_{\tau p}\Big[\Big(\frac{e^{-\epsilon\Box_{\tau p}}-I}{\epsilon}\Big)[f_\tau(\cdot,w)](z)\Big] + \epsilon\Big(\frac{I-e^{-\epsilon\Box_{\tau p}}}{\epsilon}\Big)[M^{\cdot,w}_{\tau p} f_\tau(\cdot,w)](z) \\
&= e^{-\epsilon\Box_{\tau p}}[M^{\cdot,w}_{\tau p} f_\tau(\cdot,w)](z) + \epsilon\Big[M^{z,w}_{\tau p},\Big(\frac{e^{-\epsilon\Box_{\tau p}}-I}{\epsilon}\Big)\Big][f(\cdot,w)](z)
\end{aligned}$$

By the spectral theorem, $\lim_{\epsilon\to 0}\frac{e^{-\epsilon\Box_{\tau p}}-I}{\epsilon} = \Box_{\tau p}$, and $[\Box_{\tau p}, M^{z,\xi}_{\tau p}]$ applied to a derivative of $H_{\tau p}(s,z,\xi)$ is well-controlled. Thus,

$$\lim_{\epsilon\to 0} M^{z,w}_{\tau p}\big(e^{-\epsilon\Box_{\tau p}}[H_{\tau p}(s,\cdot,w)](z)\big) = \lim_{\epsilon\to 0}\big(e^{-s\Box_{\tau p}}[M^{\cdot,w}_{\tau p} H_{\tau p}(s,\cdot,w)](z)\big) = M^{z,w}_{\tau p} H_{\tau p}(s,z,w). \tag{23}$$

By a repeated use of (23), taking the limit as $\epsilon\to 0$ in (22), we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\mathbb{C}} H_{\tau p}(s,z,\xi) U^\alpha_w X^\beta_\xi (M^{\xi,w}_{\tau p})^n H_{\tau p}(\epsilon,\xi,w)\, dA(\xi) \\
&= \lim_{\epsilon\to 0}(-1)^{|\beta|}\sum_{k=0}^{n}\sum_{j=0}^{n-k}\gamma^{n,k}_j \overline{X^\alpha_w (M^{w,z}_{\tau p})^{n-j-k} e^{-\epsilon\Box_{\tau p}}\big[\overline{r(w,\cdot,z)}^k (M^{\cdot,z}_{\tau p})^j X^\beta_\xi H_{\tau p}(s,\cdot,z)\big](w)} \\
&= \lim_{\epsilon\to 0}\Big[(-1)^{|\beta|}\sum_{k=0}^{n}\sum_{j=0}^{n-k}\gamma^{n,k}_j \overline{X^\alpha_w e^{-\epsilon\Box_{\tau p}}\big[\overline{r(w,\cdot,z)}^k (M^{\cdot,z}_{\tau p})^{n-k} X^\beta_\xi H_{\tau p}(s,\cdot,z)\big](w)} + \epsilon(OK)\Big] \\
&= \lim_{\epsilon\to 0}(-1)^{|\beta|}\sum_{k=0}^{n}\sum_{j=0}^{n-k}\gamma^{n,k}_j \overline{X^\alpha_w e^{-\epsilon\Box_{\tau p}}\big[\overline{r(w,\cdot,z)}^k (M^{\cdot,z}_{\tau p})^{n-k} X^\beta_\xi H_{\tau p}(s,\cdot,z)\big](w)}.
\end{aligned}$$

Since $\lim_{\epsilon\to 0} e^{-\epsilon\Box_{\tau p}} = I$,

$$\begin{aligned}
&\lim_{\epsilon\to 0}(-1)^{|\beta|}\sum_{k=0}^{n}\sum_{j=0}^{n-k}\gamma^{n,k}_j \overline{X^\alpha_w e^{-\epsilon\Box_{\tau p}}\big[\overline{r(w,\cdot,z)}^k (M^{\cdot,z}_{\tau p})^{n-k} X^\beta_\xi H_{\tau p}(s,\cdot,z)\big](w)} \\
&\qquad\qquad = (-1)^{|\beta|}\sum_{k=0}^{n}\sum_{j=0}^{n-k}\gamma^{n,k}_j U^\alpha_w\big[r(w,\xi,z)^k (M^{z,w}_{\tau p})^{n-k} X^\beta_\xi H_{\tau p}(s,z,\xi)\big]\Big|_{\xi=w}.
\end{aligned}$$

By Proposition 7.7 and Lemma 7.6,

$$\sum_{j=0}^{n-k}\gamma^{n,k}_j = (-1)^k\binom{n}{k}\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j} = \delta_0(n-k).$$

This means $k=n$, so

$$(-1)^{|\beta|}\sum_{k=0}^{n}\sum_{j=0}^{n-k}\gamma^{n,k}_j U^\alpha_w\big[r(w,\xi,z)^k (M^{z,w}_{\tau p})^{n-k} X^\beta_\xi H_{\tau p}(s,z,\xi)\big]\Big|_{\xi=w} = (-1)^{|\beta|} U^\alpha_w\big[r(w,\xi,z)^n X^\beta_\xi H_{\tau p}(s,z,\xi)\big]\Big|_{\xi=w}.$$

□

Remark 7.8. We have showed that the first term in (9) satisfies the bounds of Theorem 4.2 and Theorem 4.10, so we can now concentrate solely on the double integral term.

It turns out that for the remainder of this estimate, we will not need (or use) the fact that we can restrict ourselves to the case $Y^J = U^\alpha_w X^\beta_z (M^{z,w}_{\tau p})^n$.

7.3. **Cancellation Conditions for $H^s_{\tau p}$.** Unless explicitly stated, we now assume that $\tau > 0$ for the remainder of the paper. The cancellation conditions for $H^s_{\tau p}$ are proven in stages.

Lemma 7.9. *Let $n \geq 1$. If $H^s_{\tau p}$ satisfies the (m,∞)-size and cancellation conditions for $0 \leq m \leq n-1$, then $H^s_{\tau p}$ satisfies the $(n,0)$-cancellation condition.*

Proof. Fix $(s,z) \in (0,\infty)\times\mathbb{C}$. Let $\delta \leq \max\{\mu_p(z,1/\tau), s^{\frac{1}{2}}\}$ and $\varphi \in \mathcal{C}^\infty_c(D(z,\delta))$. Let $Y^J = (M^{z,w}_{\tau p})^n$. From Proposition 6.3 and Remark 7.8,

$$(M^{z,w}_{\tau p})^n H_{\tau p}(s,z,w) = \int_0^s \int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)(M^{\xi,w}_{\tau p})^k \big[\Box_{\tau p,\xi}, M^{\xi,w}_{\tau p}\big](M^{\xi,w}_{\tau p})^{n-k-1} H_{\tau p}(r,\xi,w)\, dA(\xi)\, dr.$$

However, $\big[M^{\xi,w}_{\tau p}, [\Box_{\tau p,\xi}, M^{\xi,w}_{\tau p}]\big] = -2|e(w,\xi)|^2$, so

$$\begin{aligned}(M^{\xi,w}_{\tau p})^k \big[\Box_{\tau p,\xi}, M^{\xi,w}_{\tau p}\big](M^{\xi,w}_{\tau p})^{n-k-1} &= (M^{\xi,w}_{\tau p})^{k-1} \big[\Box_{\tau p,\xi}, M^{\xi,w}_{\tau p}\big](M^{\xi,w}_{\tau p})^{n-k} - 2|e(w,\xi)|^2 (M^{\xi,w}_{\tau p})^{n-2}\\ &= \cdots = [\Box_{\tau p,\xi}, M^{\xi,w}_{\tau p}](M^{\xi,w}_{\tau p})^{n-1} - 2j|e(w,\xi)|^2 (M^{\xi,w}_{\tau p})^{n-2}\end{aligned}$$

Thus,

$$\begin{aligned}(M^{z,w}_{\tau p})^n H_{\tau p}(s,z,w) = n \int_0^s \int_{\mathbb{C}} &H_{\tau p}(s-r,z,\xi)[\Box_{\tau p,\xi}, M^{\xi,w}_{\tau p}](M^{\xi,w}_{\tau p})^{n-1} H_{\tau p}(r,\xi,w)\, dA(\xi)\, dr \\ &- n(n-1)\int_0^s \int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)|e(w,\xi)|^2 H_{\tau p}(r,\xi,w)\, dA(\xi)\, dr.\end{aligned} \tag{24}$$

We now test against a test function. To estimate $M^{z,\cdot}_{\tau p} H^s[\varphi](z)$, we start with the second integral from 24. We rewrite

$$\begin{aligned}\int_0^s \int_{\mathbb{C}}\int_{\mathbb{C}} &H_{\tau p}(s-r,z,\xi)|e(w,\xi)|^2 H_{\tau p}(r,\xi,w)\,\varphi(w)\, dA(\xi)\, dA(w)\, dr \\ &= \int_0^s \int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)(M^{\xi,w}_{\tau p})^{n-2} H^r_{\tau p}[|e(\cdot,\xi)|^2\varphi](\xi)\, dA(\xi)\, dr.\end{aligned} \tag{25}$$

We can estimate (25) with the $(n-2,0)$-cancellation condition, Corollary 6.7 and (in the case $\Delta = \mu_p(z,1/\tau)$) Lemma 6.5. We have (with a possible decrease in c),

$$\begin{aligned}&\frac{1}{\delta}\int_0^s \int_{\mathbb{C}} \frac{1}{s-r} e^{-c\frac{|z-\xi|^2}{s-r}} e^{-c(\frac{s-r}{\mu_p(z,1/\tau)^2})^\epsilon} e^{-c(\frac{s-r}{\mu_p(\xi,1/\tau)^2})^\epsilon} \Lambda(\xi,\Delta)^{n-2}\big(\||e(\cdot,\xi)|^2\varphi\|_{L^2} + \delta^2\|\Box_{\tau p}|e(\cdot,\xi)|^2\varphi\|_{L^2}\big)\, dA(\xi) dr\\ &\leq \frac{\Lambda(z,\Delta)^{n-2}}{\delta}\int_0^s \int_{\mathbb{C}} \frac{1}{s-r} e^{-c\frac{|z-\xi|^2}{s-r}} e^{-c(\frac{s-r}{\mu_p(z,1/\tau)^2})^\epsilon} e^{-c(\frac{s-r}{\mu_p(\xi,1/\tau)^2})^\epsilon}\Big(\sup_{w\in D(z,\delta)} |e(w,\xi)|^2\big[\|\varphi\|_{L^2} + \delta^2\|\Box_{\tau p}\varphi\|_{L^2}\big]\\ &+ \delta^2 \sup_{w\in D(z,\delta)} |\nabla e(w,\xi)|^2 \|\widetilde{\nabla}\varphi\|_{L^2} + \delta^2 \sup_{w\in D(z,\delta)} \big|\nabla^2|e(w,\xi)|^2\big|\|\varphi\|_{L^2}\Big)\, dA(\xi) dr.\end{aligned}$$

In the second line, we changed $\Lambda(\xi,\Delta)$ to $\Lambda(z,\Delta)$ and brought it outside of the integral. This is possible by reexpanding a^ξ_{jk} in terms of a^z_{jk} and using Lemma 6.5. If $\Delta = s^{\frac{1}{2}}$, we can apply Corollary 6.7 to attain the desired result.

The case $\Delta = \mu_p(z,1/\tau)$ requires a more delicate estimate. Note that $\Lambda(z,\Delta) \sim 1/\tau$. We bound $D^k|e(w,\xi)|^2$ for $k \leq 2$. If $\frac{1}{2}s^{\frac{1}{2}} \leq \mu_p(z,1/\tau)$ and $|\xi - z| \leq 2\mu_p(z,1/\tau)$, then for $w \in D(z,\delta)$, $|\xi - w| \lesssim \mu_p(z,1/\tau)$, so

$$|D^k e(w,\xi)|^2 \lesssim \frac{1}{\tau^2 s\delta^k}.$$

If $|z-\xi| \geq 2\mu_p(z,1/\tau)$, then $|w-\xi| \sim |z-\xi|$. Also, $\mu_p(z,1/\tau) \sim \mu_p(w,1/\tau)$ since $|z-w| \lesssim \mu_p(z,1/\tau)$. By Lemma 6.5 (use the argument of Lemma 6.5 with $\mu_p(w,1/\tau)$ and use the fact that $s^{\frac{1}{2}} \lesssim \mu_p(w,1/\tau)$),

$$e^{-c\frac{|w-\xi|^2}{s-r}} e^{-c(\frac{s-r}{\mu_p(z,1/\tau)^2})^\epsilon} e^{-c(\frac{s-r}{\mu_p(\xi,1/\tau)^2})^\epsilon} D^k|e(w,\xi)|^2 \lesssim e^{-c\frac{|w-\xi|^2}{s-r}} e^{-c(\frac{s-r}{\mu_p(z,1/\tau)^2})^\epsilon} e^{-c(\frac{s-r}{\mu_p(\xi,1/\tau)^2})^\epsilon} \frac{1}{\tau^2 s\delta^k}.$$

Then we bound (25) by

$$\frac{C_n}{\tau^n s\delta}(\|\varphi\|_{L^2} + \delta^2\|\Box_{\tau p}\varphi\|_{L^2})\int_0^s \int_{\mathbb{C}} \frac{1}{s-r} e^{-c\frac{|z-\xi|^2}{s-r}}\, dA(\xi) dr \leq \frac{C_n}{\tau^n\delta}(\|\varphi\|_{L^2} + \delta^2\|\Box_{\tau p}\varphi\|_{L^2}),$$

the desired estimate. If $\mu_p(z,1/\tau) \le \delta \le s^{\frac{1}{2}}$, the integral in (25) can be bounded as follows:

$$\begin{aligned}
&\int_{\frac{s}{2}}^{s}\int_{\mathbb{C}}\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)(M_{\tau p}^{\xi,w})^{n-2}H_{\tau p}(r,\xi,w)|e(w,\xi)|^2\varphi(w)\,dA(\xi)dA(w)dr\\
&\lesssim \int_{\frac{s}{2}}^{s}\int_{\mathbb{C}}\int_{\mathbb{C}} \frac{1}{s-r}e^{-c\frac{|z-\xi|^2}{s-r}}\frac{1}{s\tau^{n-2}}e^{-c\frac{|\xi-w|^2}{s}}e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^\epsilon}e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^\epsilon}|e(w,\xi)|^2|\varphi(w)|\,dA(\xi)dA(w)dr\\
&\lesssim \int_{\frac{s}{2}}^{s}\frac{1}{\tau^n s\delta}\int_{\mathbb{C}}|\varphi(w)|\,dA(w) \lesssim \frac{1}{\tau^n}\|\varphi\|_{L^2}.
\end{aligned}$$

The integral from 0 to $\frac{s}{2}$ is estimated similarly.

To estimate the first integral in (24) tested against a test function, we concentrate on each term of $[\Box_{\tau p,\xi}, M_{\tau p}^{\xi,w}]$ from Proposition 6.4 separately. They are handled analogously, and we will only discuss the $Z_{\tau p,\xi}(M_{\tau p}^{\xi,w})^{n-1}$ term. Although the operator $Z_{\tau p,\xi}(M_{\tau p}^{\xi,w})^{n-1}$ is an operator of order $(n-1,1)$ and hence under control, if we naively applied the $(n-1,1)$-cancellation condition, the result would not satisfy the $(n,0)$-cancellation condition estimate. Derivatives of too high an order would appear. Instead, we bring the $Z_{\tau p,\xi}$ onto the $H_{\tau p}(s-r,z,\xi)$ term.

$$\begin{aligned}
&\int_0^s\int_{\mathbb{C}}\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)e(w,\xi)Z_{\tau p,\xi}(M_{\tau p}^{\xi,w})^{n-1}H_{\tau p}(r,\xi,w)\,\varphi(w)\,dA(w)dA(\xi)dr\\
&\qquad= -\int_0^s\int_{\mathbb{C}} W_{\tau p,\xi}H_{\tau p}(s-r,z,\xi)\,(M_{\tau p}^{\xi,w})^{n-1}H_{\tau p}^r[e(\cdot,\xi)\varphi](\xi)\,dA(\xi)dr\\
&\qquad\quad -\int_0^s\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)\,(M_{\tau p}^{\xi,w})^{n-1}H_{\tau p}^r\Big[\frac{\partial e(\cdot,\xi)}{\partial\xi}\varphi\Big](\xi)\,dA(\xi)dr. \qquad (26)
\end{aligned}$$

The argument for (26) is essentially a repeat of the argument for (25). Also, the $n=1$ case is handled easily using the arguments for the $n\ge 2$ case (in fact, the proof of the $n=1$ case is contained in the proof of the $n=2$ case – no commutators are needed). □

From [6], we have

Lemma 7.10. *If $|z-\zeta| > \mu_p(\zeta,1/\tau)$, then there exist constants C, M, and $\delta = \frac{2}{\deg p}$ so that*

$$\frac{\mu_p(\zeta,1/\tau)}{\mu_p(z,1/\tau)} \le C\Big(\frac{|z-\zeta|}{\mu_p(\zeta,1/\tau)}\Big)^M \qquad \text{and} \qquad \frac{\mu_p(z,1/\tau)}{\mu_p(\zeta,1/\tau)} \le C\Big(\frac{|z-\zeta|}{\mu_p(\zeta,1/\tau)}\Big)^{1-\delta}.$$

In [6], Christ only finds the exists of $\delta > 0$ so that the second inequality holds. However, using reverse Hölder classes and the techniques of [22], we can explicitly find δ. We omit the computation because we will only use that $\delta > 0$; a quantitative estimate of δ is not necessary for our work.

Lemma 7.11. *Let $n \ge 1$ and $\ell \ge 1$ be integers. If $H_{\tau p}^s$ satisfies the (j,∞)-size and cancellation conditions for $0 \le j \le n-1$ and the (n,k)-size and cancellation conditions for $0 \le k \le \ell-1$, then $H_{\tau p}^s$ satisfies the (n,ℓ)-cancellation conditions.*

Proof. Let $Y^J \in (n,\ell)$ and $\varphi \in \mathcal{C}_c^\infty(D(z,\delta))$ where $\delta \le \max\{\mu_p(z,1/\tau), s^{1/2}\}$. We start by reducing $Y^J H_{\tau p}^s[\varphi]$ into integrals for which our inductive hypothesis is valid.

By Proposition 6.3, we must estimate

$$\Big|\int_0^s\int_{\mathbb{C}}\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)\Big(\prod_{\imath=0}^{k}Y_{|J|-\imath}\Big)\big[\Box_{\tau p,\xi}, Y_{|J|-k-1}\big]Y^{J-k-2}H_{\tau p}(r,\xi,w)\,\varphi(w)\,dA(w)dA(\xi)dr\Big|.$$

Let $\Big(\prod_{\imath=0}^{k}Y_{|J|-\imath}\Big) = Y^K$ (so $|K| = k+1$). The commutator $\big[\Box_{\tau p,\xi}, Y_{|J|-k-1}\big]$ is nonzero only if $Y_{|J|-k-1} = M_{\tau p}^{z,w}$ or X_ξ. If $Y_{|J|-k-1} = M_{\tau p}^{z,w}$, by Proposition 6.4, the integral to estimate is

$$\Big|\int_0^s\int_{\mathbb{C}}\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)Y^K\Big(-\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar\xi} - e(w,\xi)Z_{\tau p,\xi} + \overline{e(w,\xi)}\overline{Z}_{\tau p}\Big)Y^{J-k-2}H_{\tau p}(r,\xi,w)\,\varphi(w)\,dA(w)dA(\xi)dr\Big|. \qquad (27)$$

When $Y_{|J|-k-1} = X_\xi$, the integral to bound can be written as

$$\left|\tau\int_0^s\int_{\mathbb{C}}\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)Y^K\Big(c_1\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar\xi}Z_{\tau p,\xi}+c_2\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar\xi}\overline{Z}_{\tau p,\xi}+c_3\frac{\partial^3 p(\xi)}{\partial\xi\partial\bar\xi^2}\Big)Y^{J-k-2}H_{\tau p}(r,\xi,w)\,\varphi(w)\,dA(w)dA(\xi)dr\right| \tag{28}$$

where c_i, $i=1,2,3$ are constants that depend on whether $X_\xi = \overline{Z}_{\tau p,\xi}$, $Z_{\tau p,\xi}$, etc. In the case $Y_{|J|-k-1} = M^{z,w}_{\tau p}$, the operator $Y^K Y^{J-k-2} \in (n-1,\ell)$ so even after factoring $\big(-\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar\xi}+e(w,\xi)Z_{\tau p,\xi}+\overline{e(w,\xi)}\overline{Z}_{\tau p}\big)$ into the derivative, the derivative is at worst an $(n-1,\ell+1)$-derivative and covered by the induction hypothesis. In the second case, $Y^KY^{J-k-2}\in(n,\ell-1)$, so after taking $\big(c_1\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar\xi}Z_{\tau p,\xi}+c_2\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar\xi}\overline{Z}_{\tau p,\xi}+c_3\frac{\partial^3 p(\xi)}{\partial\xi\partial\bar\xi^2}\big)$ into account, the derivative can be an (n,ℓ)-derivative. Thus, we cannot immediately use the induction hypothesis. We can, however, integrate by parts to bring a ξ-derivative onto the $H_{\tau p}(s-r,z,\xi)$-term and use our size and cancellation conditions to estimate the integral.

Fortunately, the estimations of (27) and (28) can be done in a similar fashion, so we only present the case $Y_{|J|-k-1} = M^{\xi,w}_{\tau p}$. In (27), the commutator $[\Box_{\tau p,\xi}, M^{\xi,w}_{\tau p}]$ creates a sum of three terms. Each of these terms can be estimated with the same techniques, so we only demonstrate the estimate of

$$\begin{aligned}&\int_0^s\int_{\mathbb{C}}\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)Y^K\big(e(w,\xi)Z_{\tau p,\xi}Y^{J-k-2}H_{\tau p}(r,\xi,w)\big)\varphi(w)\,dA(w)dA(\xi)dr\\
&=\sum_{|K_1|+|K_2|=k+1} c_{K_1,K_2}\int_0^s\int_{\mathbb{C}}\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)D^{K_1}_\xi e(w,\xi)Y^{K_2}Z_{\tau p,\xi}Y^{J-k-2}H_{\tau p}(r,\xi,w)\varphi(w)\,dA(w)dA(\xi)dr.\end{aligned}$$

To integrate by parts, observe that we can write $Y^{K_2}Z_{\tau p,\xi} = Y^{\alpha_1}X_\xi Y^{\alpha_2}$ where Y^{α_1} is composed only of $M^{\xi,w}_{\tau p}$, $\overline{W}_{\tau p,w}$, and $W_{\tau p,w}$. This means X_ξ is the first ξ-derivative. Of course, X_ξ commutes with $\overline{W}_{\tau p,w}$ and $W_{\tau p,w}$. Also, $[M^{\xi,w}_{\tau p}, X_\xi] = e(w,\xi)$ or $\overline{e(w,\xi)}$, we can commute X_ξ by $M^{\xi,w}_{\tau p}$ with an error of $e(w,\xi)$. Thus, with the convention that $Y_{|J|+1}=1$,

$$\begin{aligned}Y^{\alpha_1}X_\xi Y^{\alpha_2} &= Y_{|J|}\cdots Y_{|J|-|\alpha_1|+1}X_\xi Y^{\alpha_2}\\
&= X_\xi Y_{|J|}\cdots Y_{|J|-|\alpha_1|+1}Y^{\alpha_2} + \sum_{n=0}^{|\alpha_1|-1} Y_{|J|}\cdots Y_{|J|-n+1}\big[Y_{|J|-n},X_\xi\big]Y_{|J|-n-1}\cdots Y_{|J|-|\alpha_1|+1}Y^{\alpha_2}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\int_0^s\int_{\mathbb{C}}\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)D^{K_1}_\xi e(w,\xi)Y^{K_2}Z_{\tau p,\xi}Y^{J-k-2}H_{\tau p}(r,\xi,w)\varphi(w)\,dA(w)dA(\xi)dr\\
&=\int_0^s\int_{\mathbb{C}}\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)D^{K_1}_\xi e(w,\xi)X_\xi Y_{|J|}\cdots Y_{|J|-|\alpha_1|+1}Y^{\alpha_2}Y^{J-k-2}H_{\tau p}(r,\xi,w)\,\varphi(w)\,dA(w)dA(\xi)dr\\
&+\sum_{\jmath=0}^{|\alpha_1|-1}\int_0^s\int_{\mathbb{C}}\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)D^{K_1}_\xi e(w,\xi)Y_{|J|}\cdots Y_{|J|-\jmath+1}\big[Y_{|J|-\jmath},X_\xi\big]\times\\
&\qquad\qquad Y_{|J|-\jmath-1}\cdots Y_{|J|-|\alpha_1|+1}Y^{\alpha_2}Y^{J-k-2}H_{\tau p}(r,\xi,w)\,\varphi(w)\,dA(w)dA(\xi)dr.\end{aligned}$$

The integrals in sum can be handled using the size and cancellation conditions from the induction hypotheses in the same manner as the first integral (after we integrate by parts in the first integral). We only show the computation for the first (and most difficult) integral.

$$\begin{aligned}&\left|\int_0^s\int_{\mathbb{C}}\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)D^{K_1}_\xi e(w,\xi)X_\xi Y_{|J|}\cdots Y_{|J|-|\alpha_1|+1}Y^{\alpha_2}Y^{J-k-2}H_{\tau p}(r,\xi,w)\,\varphi(w)\,dA(w)dA(\xi)dr\right|\\
&=\Bigg|\int_0^s\int_{\mathbb{C}}\int_{\mathbb{C}}\big(X^\#_\xi H_{\tau p}(s-r,z,\xi)D^{K_1}_\xi e(w,\xi)+H_{\tau p}(s-r,z,\xi)D_\xi D^{K_1}_\xi e(w,\xi)\big)\times\\
&\qquad\qquad Y_{|J|}\cdots Y_{|J|-|\alpha_1|+1}Y^{\alpha_2}Y^{J-k-2}H_{\tau p}(r,\xi,w)\,\varphi(w)\,dA(w)dA(\xi)dr\Bigg|\end{aligned}$$

The two integrals can be estimated with the same size and cancellation condition argument, and we will show only the estimate of

$$\left| \int_0^s \int_{\mathbb{C}} \int_{\mathbb{C}} X_\xi^{\#} H_{\tau p}(s-r,z,\xi) D_\xi^{K_1} e(w,\xi) Y_{|J|} \cdots Y_{|J|-|\alpha_1|+1} Y^{\alpha_2} Y^{J-k-2} H_{\tau p}(r,\xi,w)\, \varphi(w)\, dA(w) dA(\xi) dr \right|. \tag{29}$$

Note that $Y_{|J|} \cdots Y_{|J|-|\alpha_1|+1} Y^{\alpha_2} Y^{J-k-2} \in (n-1, \ell-|K_1|)$. (If we had chosen $Y_{|J|-k-1} = X_\xi$, we would have a similar integral with a derivative of $\triangle p$ replacing $e(w,\xi)$ and $Y_{|J|} \cdots Y_{|J|-|\alpha_1|+1} Y^{\alpha_2} Y^{J-k-2} \in (n, \ell-1-|K_1|)$). Our induction hypothesis applies. We break the s-integral in (29) into two pieces and estimate each piece separately.

We show the argument for ℓ and $|K_1|$ even, but the cases when at least of ℓ and $\ell - |K_1|$ are odd is done similarly. For $0 \leq r \leq s/2$, $(s-r) \sim s$, so

$$\begin{aligned}
&\left| \int_0^{s/2} \int_{\mathbb{C}} \int_{\mathbb{C}} X_\xi^{\#} H_{\tau p}(s-r,z,\xi) D_\xi^{K_1} e(w,\xi) Y_{|J|} \cdots Y_{|J|-|\alpha_1|+1} Y^{\alpha_2} Y^{J-k-2} H_{\tau p}(r,\xi,w)\, \varphi(w)\, dA(w) dA(\xi) dr \right| \\
&= \left| \int_0^{s/2} \int_{\mathbb{C}} X_\xi^{\#} H_{\tau p}(s-r,z,\xi) Y_{|J|} \cdots Y_{|J|-|\alpha_1|+1} Y^{\alpha_2} Y^{J-k-2} H_{\tau p}^r \big[D_\xi^{K_1} e(\cdot,\xi)\varphi \big](\xi)\, dA(\xi) dr \right| \\
&\lesssim \int_0^{s/2} \int_{\mathbb{C}} \frac{1}{s^{3/2}} e^{-c\frac{|z-\xi|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(\xi,1/\tau)^2}} \frac{\Lambda(\xi,\Delta)^{n-1}}{\delta} \big(\| \Box_{\tau p}^{\frac12(\ell-|K_1|)} \big(D_\xi^{K_1} e(\cdot,\xi)\varphi \big) \|_{L^2} \\
&\qquad + \delta^2 \| \Box_{\tau p}^{\frac12(\ell-|K_1|)+1} \big(D_\xi^{K_1} e(\cdot,\xi)\varphi \big) \|_{L^2} \big)\, dA(\xi) dr.
\end{aligned}$$

The two terms are handled similarly, and we show the estimate of

$$\int_0^{s/2} \int_{\mathbb{C}} \frac{1}{s^{3/2}} e^{-c\frac{|z-\xi|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(\xi,1/\tau)^2}} \frac{\Lambda(\xi,\Delta)^{n-1}}{\delta} \| \Box_{\tau p,w}^{\frac12(\ell-|K_1|)} \big(D_\xi^{K_1} e(\cdot,\xi)\varphi \big) \|_{L^2}\, dA(\xi) dr.$$

Since

$$\Box_{\tau p,w}^{\frac12(\ell-|K_1|)} \big(D_\xi^{K_1} e(\cdot,\xi)\varphi \big) = \sum_{|\gamma_1|+\gamma_2|=\ell-|K_1|} c_{\gamma_1,\gamma_2} D^{\gamma_1} D^{K_1} e(w,\xi) X^{\gamma_2} \varphi(w),$$

it follows that

$$\begin{aligned}
&\int_0^{s/2} \int_{\mathbb{C}} \frac{1}{s^{3/2}} e^{-c\frac{|z-\xi|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(\xi,1/\tau)^2}} \frac{\Lambda(\xi,\Delta)^{n-1}}{\delta} \| \Box_{\tau p}^{\frac12(\ell-|K_1|)} \big(D_\xi^{K_1} e(\cdot,\xi)\varphi \big) \|_{L^2}\, dA(\xi) dr \\
&\lesssim \sum_{|\gamma_1|+|\gamma_2|=\ell-|K_1|} \int_0^{s/2} \int_{\mathbb{C}} \frac{1}{s^{3/2}} e^{-c\frac{|z-\xi|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(\xi,1/\tau)^2}} \frac{\Lambda(\xi,\Delta)^{n-1}}{\delta} \| D^{\gamma_1} D^{K_1} e(\cdot,\xi) X^{\gamma_2} \varphi(\cdot) \|_{L^2}\, dA(\xi) dr
\end{aligned} \tag{30}$$

To estimate $\Lambda(x,\Delta)$ and $D^{\gamma_1} D^{K_1} e(w,\xi)$, we turn Lemma 6.5, Corollary 6.7, the proofs of these two results, and (14). With a decrease in c, we can bound

$$e^{-c\frac{|z-\xi|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(\xi,1/\tau)^2}} \Lambda(\xi,\Delta)^{n-1} \lesssim e^{-c\frac{|z-\xi|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(\xi,1/\tau)^2}} \Lambda(z,\Delta)^{n-1}.$$

Also, again with a decrease in c, since $|z-w| \leq \delta$,

$$\begin{aligned}
&e^{-c\frac{|z-\xi|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(\xi,1/\tau)^2}} |D^{\gamma_1} D^{K_1} e(\cdot,\xi)| \\
&\lesssim e^{-c\frac{|z-\xi|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(\xi,1/\tau)^2}} \min\{\Lambda(z,s^{1/2}) s^{-\frac12(1+|K_1|+|\gamma_1|)}, \tau^{-1}\mu_p(\xi,1/\tau)^{-(1+|K_1|+|\gamma_1|)} \\
&\lesssim e^{-c\frac{|z-\xi|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(\xi,1/\tau)^2}} \Lambda(z,\Delta) \min\{ s^{-\frac12(1+|K_1|+|\gamma_1|)}, \mu_p(z,1/\tau)^{-(1+|K_1|+|\gamma_1|)} \}.
\end{aligned}$$

In this previous string of inequalities, the first estimate uses Lemma 6.5 while the second inequality is justified with Lemma 7.10 and a reduction of c in the exponent to control terms of the form $\big(|z-\xi| / \mu_p(z,1/\tau) \big)^M$.

Thus, choosing an arbitrary term from (30), we estimate

$$\int_0^{s/2}\int_{\mathbb{C}} \frac{1}{s^{3/2}} e^{-c\frac{|z-\xi|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(\xi,1/\tau)^2}} \frac{\Lambda(\xi,\Delta)^{n-1}}{\delta} \|D^{\gamma_1} D^{K_1} e(\cdot,\xi) X^{\gamma_2}\varphi(\cdot)\|_{L^2}\, dA(\xi) dr$$

$$\lesssim \|X^{\gamma_2}\varphi(\cdot)\|_{L^2} \int_0^{s/2}\int_{\mathbb{C}} \frac{1}{s^{3/2}} e^{-c\frac{|z-\xi|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(\xi,1/\tau)^2}} \frac{\Lambda(\xi,\Delta)^{n-1}}{\delta} \sup_{w\in \operatorname{supp}\varphi} |D^{\gamma_1} D^{K_1} e(w,\xi)|\, dA(\xi) dr \tag{31}$$

$$\lesssim \|X^{\gamma_2}\varphi(\cdot)\|_{L^2} \frac{\Lambda(z,\Delta)^n}{\delta \max\{s^{\frac12},\mu_p(z,1/\tau)\}^{1+|K_1|+|\gamma_1|}} \int_0^{s/2}\int_{\mathbb{C}} \frac{1}{s^{3/2}} e^{-c\frac{|z-\xi|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(\xi,1/\tau)^2}}\, dA(\xi) dr$$

$$\lesssim \|X^{\gamma_2}\varphi(\cdot)\|_{L^2} \frac{\Lambda(z,\Delta)^n}{\delta \max\{s^{\frac12},\mu_p(z,1/\tau)\}^{1+|K_1|+|\gamma_1|}} s^{1/2}. \tag{32}$$

From [21], we know $\|\varphi\|_{L^2} \le \sqrt{2}\delta \|X_j\varphi\|_{L^2}$ for $j=1,2$. Also, from [20], if $X^\alpha \in (0,\ell)$, then $\|X^\alpha\varphi\|_{L^2} \sim \|\Box_{\tau p}^{\ell/2}\varphi\|_{L^2}$. Thus, since $|\gamma_1|+|\gamma_2| = \ell - |K_1|$,

$$\|X^{\gamma_2}\varphi(\cdot)\|_{L^2} \frac{\Lambda(z,\Delta)^n}{\delta \max\{s^{\frac12},\mu_p(z,1/\tau)\}^{1+|K_1|+|\gamma_1|}} s^{1/2} \lesssim \frac{\Lambda(z,\Delta)^n}{\delta} \|\Box_{\tau p}^{\ell/2}\varphi\|_{L^2},$$

the desired estimate.

We have one integral remaining to estimate. If $\Delta = \sqrt{s}$, then we can use the integral estimate as the $0 \le r \le s/2$ case. We can follow the estimate line by line, except for two differences. First, we cannot replace $s-r$ with s. However, this is not an issue $e^{-c\frac{|z-\xi|^2}{s-r}} \le e^{-c\frac{|z-\xi|^2}{s}}$, so the the use of Corollary 6.7 to bound $e^{-c\frac{|z-\xi|^2}{s-r}}|D^{\gamma_1}D^{K_1}e(w,\xi)|$ remains unchanged. To bound $e^{-c\frac{|z-\xi|^2}{s-r}}\Lambda(\xi,\Delta)^{n-1} \lesssim e^{-c\frac{|z-\xi|^2}{s-r}}\Lambda(z,\Delta)^{n-1}$, we have (using the arguments of Lemma 6.5 and (15))

$$\begin{aligned} e^{-c\frac{|z-\xi|^2}{s-r}}\Lambda(\xi,\sqrt{s}) &= e^{-c\frac{|z-\xi|^2}{s-r}} \sum_{j,k\ge 1} |a^\xi_{jk}| s^{(j+k)/2} \lesssim e^{-c\frac{|z-\xi|^2}{s-r}} \sum_{j,k\ge1} \sum_{\substack{\alpha\ge j\\ \beta\ge k}} |a^z_{\alpha\beta}||\xi-z|^{\alpha+\beta-j-k} s^{(j+k)/2} \\ &\le e^{-c\frac{|z-\xi|^2}{s-r}} \sum_{j,k\ge1} \sum_{\substack{\alpha\ge j\\ \beta\ge k}} |a^z_{\alpha\beta}||\xi-z|^{\alpha+\beta-j-k} s^{(j+k)/2} \frac{s^{(\alpha+\beta-j-k)2}}{|\xi-z|^{\alpha+\beta-j-k}} = e^{-c\frac{|z-\xi|^2}{s-r}}\Lambda(z,\sqrt{s}). \end{aligned}$$

As usual, the bound comes with a price of a decrease in c. Last, the line of argument in (31) proceeds line by line, replacing $s-r$ with s. Since $\Delta = \{\mu_p(z,1/\tau),\sqrt{s}\}$, $\max\{s^{\frac12},\mu_p(z,1/\tau)\} = \mu_p(z,1/\tau)$.

Thus, the final integral to estimate is from $s/2$ to s in the case that $\Delta = \mu_p(z,1/\tau)$. In this case, $\delta \le \sqrt{s}$ and $\Lambda(z,\Delta) \sim 1/\tau$. We use size estimates to bound the integral. Using Lemma 6.5, we estimate

$$\left| \int_{s/2}^s \int_{\mathbb{C}}\int_{\mathbb{C}} X^\#_\xi H_{\tau p}(s-r,z,\xi) D^{K_1}_\xi e(w,\xi) Y_{|J|}\cdots Y_{|J|-|\alpha_1|+1} Y^{\alpha_2} Y^{J-k-2} H_{\tau p}(r,\xi,w)\,\varphi(w)\, dA(w) dA(\xi) dr \right|$$

$$\lesssim \int_{\mathbb{C}} |\varphi(w)| \int_{s/2}^s \int_{\mathbb{C}} \frac{1}{(s-r)^{3/2}} e^{-c\frac{|\xi-z|^2}{s-r}} |D^{K_1} e(w,\xi)| \frac{e^{-c\frac{|\xi-w|^2}{s}}}{\tau^{n-1} s^{\frac12(1+\ell-|K_1|)}} e^{-c(\frac{s}{\mu_p(\xi,1/\tau)^2})^\epsilon} e^{-c(\frac{s}{\mu_p(w,1/\tau)^2})^\epsilon}\, dA(\xi)\, dr\, dA(w)$$

$$\lesssim \int_{\mathbb{C}} |\varphi(w)| \int_{s/2}^s \int_{\mathbb{C}} \frac{1}{(s-r)^{3/2}} e^{-c\frac{|\xi-z|^2}{s-r}} \frac{s^{|K_1|/2}}{\mu_p(w,1/\tau)^{|K_1|+1}} \frac{e^{-c\frac{|\xi-w|^2}{s}}}{\tau^n s^{\frac12(1+\ell)}} e^{-c(\frac{s}{\mu_p(\xi,1/\tau)^2})^\epsilon} e^{-c(\frac{s}{\mu_p(w,1/\tau)^2})^\epsilon}\, dA(\xi)\, dr\, dA(w)$$

$$\lesssim \frac{1}{\tau^n\delta^{1+\ell}} \int_{\mathbb{C}} |\varphi(w)|\, dA(w) \le \frac{1}{\tau^n\delta^{1+\ell}} \delta^{1/2} \|\varphi_w\|_{L^2} \lesssim \frac{1}{\tau^n\delta} \|\Box_{\tau p}^{\ell/2}\varphi\|_{L^2}.$$

□

7.4. (n,ℓ)**-size estimates for** $H_{\tau p}(s,z,w)$**.**

Proposition 7.12. *Fix (n,ℓ), $0<n<\infty$ and $0\le \ell<\infty$. If $H_{\tau p}(s,z,w)$ satisfies the (n',∞)-size and cancellation conditions for $0\le n'<n$ and (n,ℓ')-size and cancellation conditions for $0\le \ell'<\ell$, then $H_{\tau p}(s,z,w)$ satisfies (n,ℓ)-size conditions.*

Proof. As above, to estimate $Y^J H_{\tau p}(s,z,w)$ for $J\in(n,\ell)$, it suffices to estimate

$$\int_0^s\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)\sum_{k=0}^{|J|-2}\Big(\prod_{\imath=0}^{k} Y_{|J|-\imath}\Big)[\Box_{\tau p,\xi},Y_{|J|-k-2}]Y^{J-k-2}H_{\tau p}(r,\xi,w)\,dA(\xi)dr.$$

Also, by the conjugate symmetry of $H_{\tau p}(s,z,w)$, i.e., $H_{\tau p}(s,z,w)=\overline{H_{\tau p}(s,w,z)}$, it is enough to obtain the bound

$$\frac{\Lambda(z,\Delta)^n}{s^{1+\frac12|\alpha|}}e^{-c\frac{|z-w|^2}{s}}e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^\epsilon}$$

for some $\epsilon>0$.

We will estimate the integral for a fixed k. Let $Y^K=\prod_{\imath=0}^k Y_{|J|-\imath}$ (so $|K|=k+1$). The are three cases to consider: $Y_{J-k-2}=M^{\xi,w}_{\tau p}$, $\overline{Z}_{\tau p,\xi}$, or $Z_{\tau p,\xi}$. First, assume $Y_{J-k-2}=M^{\xi,w}_{\tau p}$. In this case, we must estimate

$$\int_0^s\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)Y^K\Big(-\frac{\partial^2 p(\xi)}{\partial\xi\partial\bar\xi}-e(w,\xi)Z_{\tau p,\xi}+\overline{e(w,\xi)Z}_{\tau p,\xi}\Big)Y^{J-k-2}H_{\tau p}(r,\xi,w)\,dA(\xi)dr.$$

All three terms are estimated similarly, so we only show the estimate for the $e(w,\xi)Z_{\tau p,\xi}$ term. Note that $Y^K Z_{\tau p,\xi}Y^{J-k-2}\in(n-1,\ell+1)$, so we can apply size and cancellation conditions. We can write

$$Y^K\big[e(w,\xi)Z_{\tau p,\xi}Y^{J-k-2}H_{\tau p}(r,\xi,w)\big]=\sum_{|K_1|+|K_2|=K}c_{K_1,K_2}D^{K_1}_\xi e(w,\xi)Y^{K_2}Z_{\tau p,\xi}Y^{J-k-2}H_{\tau p}(r,\xi,w).$$

It is enough to bound

$$\left|\int_0^s\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)D^{K_1}_\xi e(w,\xi)Y^{K_2}Z_{\tau p,\xi}Y^{J-k-2}H_{\tau p}(r,\xi,w)\,dA(\xi)dr\right|. \tag{33}$$

If $\frac{s}{2}\le r\lesssim s$, then $r\sim s$, so by size estimates, Lemma 6.5, and Corollary 6.7,

$$\begin{aligned}
&\Big|\int_{\frac s2}^s\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)D^{K_1}_\xi e(w,\xi)Y^{K_2}Z_{\tau p,\xi}Y^{J-k-2}H_{\tau p}(r,\xi,w)\,dA(\xi)dr\Big|\\
&\lesssim\int_{\frac s2}^s\int_{\mathbb{C}}\frac{1}{s-r}e^{-c\frac{|z-\xi|^2}{s-r}}|D^{K_1}_\xi e(w,\xi)|\frac{e^{-c\frac{|w-\xi|^2}{s}}e^{-c(\frac{s}{\mu_p(w,1/\tau)^2})^\epsilon}e^{-c(\frac{s}{\mu_p(\xi,1/\tau)^2})^\epsilon}\Lambda(\xi,s^{1/2})^{n-1}}{s^{1+\frac12(|K_2|+1+|J|-k-2-(n-1))}}\,dA(\xi)dr\\
&\lesssim\frac{\Lambda(z,\Delta)^n}{s^{1+\frac12(|K_1|+1+\ell+|K_2|-k)}}e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^\epsilon}\int_{\frac s2}^s\int_{\mathbb{C}}\frac{1}{s-r}e^{-c\frac{|z-\xi|^2}{s-r}}e^{-c\frac{|w-\xi|^2}{s}}\,dA(\xi)dr.
\end{aligned}$$

Note that if $|z-\xi|\le|w-\xi|$, then $|w-\xi|\ge\frac12|z-w|$, and if $|z-\xi|\ge|w-\xi|$, then $|z-\xi|\ge\frac12|z-w|$. Thus, with a slight decrease in c,

$$\begin{aligned}
&\frac{\Lambda(z,\Delta)^n}{s^{2+\frac{\ell}{2}}}e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^\epsilon}\int_{\frac s2}^s\int_{\mathbb{C}}\frac{1}{s-r}e^{-c\frac{|z-\xi|^2}{s-r}}e^{-c\frac{|w-\xi|^2}{s}}\,dA(\xi)dr\\
&\lesssim\frac{\Lambda(z,\Delta)^n}{s^{2+\frac{\ell}{2}}}e^{-c\frac{|z-w|^2}{s}}e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^\epsilon}\int_{\frac s2}^s\int_{\mathbb{C}}\frac{1}{s-r}e^{-c\frac{|z-\xi|^2}{s-r}}\,dA(\xi)dr\le\frac{\Lambda(z,\Delta)^n}{s^{1+\frac{\ell}{2}}}e^{-c\frac{|z-w|^2}{s}}e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^\epsilon},
\end{aligned}$$

the desired estimate. The estimate for $0\le r\le s/2$ is more delicate. Let $\delta=\frac12\min\{\mu_p(w,1/\tau),s^{\frac12}\}$ and $\varphi_w\in C^\infty_c(\mathbb{C})$ where $\operatorname{supp}\varphi_w\subset D(w,2\delta)$. Let $\varphi_w\equiv1$ on $D(w,\frac12\delta)$, $0\le\varphi_w\le1$, and $|\nabla^\beta\varphi_w|\le\frac{c_k}{\delta^{|\beta|}}$. The first integral we estimate is

$$\begin{aligned}
&\int_0^{\frac s2}\int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi)D^{K_1}e(w,\xi)Y^{K_2}Z_{\tau p,\xi}Y^{J-k-2}H_{\tau p}(r,\xi,w)\varphi_w(\xi)\,dA(\xi)dr\\
&\qquad=\int_0^{\frac s2}Y^{K_2}Z_{\tau p,\xi}Y^{J-k-2}\overline{H^r_{\tau p}}\big[H_{\tau p}(s-r,z,\cdot)D^{K_1}e(w,\cdot)\varphi_w\big](w)\,dr.
\end{aligned}$$

$Y^{K_2} Z_{\tau p,\xi} Y^{J-k-2} \in (n-1, |K_2|+1+|J|-k-2-(n-1))$ and $|K_2|+1+|J|-k-2-(n-1) = \ell-|K_1|+1$. We can assume that $\ell - |K_1| + 1$ is even since the odd case is handled analogously. By the $(n-1, \ell-|K_1|+1)$-cancellation condition and using the fact that $s - r \sim s$,

$$\begin{aligned}
&\Big| \int_0^{\frac{s}{2}} Y^{K_2} Z_{\tau p,\xi} Y^{J-k-2} \overline{H^r_{\tau p}} \big[H_{\tau p}(s-r,z,\cdot) D^{K_1} e(w,\cdot)\varphi_w \big](w)\, dr \Big| \\
&\leq c_n \frac{\Lambda(z,\Delta)^{n-1}}{\delta} \int_0^{\frac{s}{2}} \Big(\big\| (\Box^{\#}_{\tau p,\xi})^{\frac{1}{2}(\ell-|K_1|+1)} \big(H_{\tau p}(s-r,z,\cdot) D^{K_1}_\xi e(w,\cdot)\varphi \big) \big\|_{L^2(\mathbb{C})} \\
&\qquad\qquad + \delta^2 \big\| (\Box^{\#}_{\tau p,\xi})^{\frac{1}{2}(\ell-|K_1|+1)+1} \big(H_{\tau p}(s-r,z,\cdot) D^{K_1}_\xi e(w,\cdot)\varphi \big) \big\|_{L^2(\mathbb{C})} \Big)\, dr
\end{aligned}$$

Since the two terms can be estimated similarly, we estimate the first term. Since $|\xi - w| < \frac{1}{2}\mu_p(w,1/\tau)$, $\mu_p(w,1/\tau) \sim \mu_p(\xi,1/\tau)$ and $|e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} D^\beta e(w,\xi)| \lesssim e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} \frac{1}{\tau s^{\frac{1}{2}(1+|\beta|)}}$. Also, since $\delta < \frac{1}{2}s^{1/2}$, $\frac{|z-\xi|}{s-r} \sim \frac{|z-w|}{s}$. Thus, by Lemma 6.5 and Corollary 6.7,

$$\begin{aligned}
&\frac{\Lambda(z,\Delta)^{n-1}}{\delta} \int_0^{\frac{s}{2}} \big\| (\Box^{\#}_{\tau p,\xi})^{\frac{1}{2}(\ell-|K_1|+1)} \big(H_{\tau p}(s-r,z,\cdot) D^{K_1}_\xi e(w,\cdot)\varphi \big) \big\|_{L^2(\mathbb{C})}\, dr \\
&\frac{\Lambda(z,\Delta)^{n-1}}{\delta} \int_0^{\frac{s}{2}} \sum_{|\alpha_1|+|\alpha_2|+|\alpha_3| = \ell+1-|K_1|} \big\| U^{\alpha_1}_\xi H_{\tau p}(s-r,z,\cdot) D^{\alpha_2}_\xi D^{K_1}_\xi e(w,\cdot) D^{\alpha_3}_\xi \varphi_w \big\|_{L^2(\mathbb{C})}\, dr \\
&\lesssim \frac{\Lambda(z,\Delta)^{n-1}}{\delta} \int_0^{\frac{s}{2}} \Big\| \frac{1}{s^{1+\frac{1}{2}|\alpha_1|}} e^{-c\frac{|z-\xi|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(\xi,1/\tau)^2}} |D^{|\alpha_2|+|K_1|} e(w,\xi)| \Big\|_{L^2(\operatorname{supp}\varphi_w)} \frac{1}{\delta^{|\alpha_3|}}\, dr \\
&\lesssim \frac{\Lambda(z,\Delta)^n}{s^{1+\frac{1}{2}(|\alpha_1|+|\alpha_2|+|K_1|+1)}} \int_0^{\frac{s}{2}} e^{-c\frac{|z-w|^2}{s}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} \frac{1}{\delta^{|\alpha_3|}}\, dr \\
&\lesssim e^{-c\frac{|z-w|^2}{s}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} \frac{\Lambda(z,\Delta)^n}{s^{\frac{1}{2}(|\alpha_1|+|\alpha_2|+|K_1|+1+|\alpha_3|)}} = \frac{\Lambda(z,\Delta)^n}{s^{1+\frac{1}{2}\ell}} e^{-c\frac{|z-w|^2}{s}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}}.
\end{aligned}$$

The lemma will be proved once we estimate

$$\int_0^{\frac{s}{2}} \int_{\mathbb{C}} H_{\tau p}(s-r,z,\xi) D^{K_1} e(w,\xi) Y^{K_2} Z_{\tau p,\xi} Y^{J-k-2} H_{\tau p}(r,\xi,w)(1-\varphi_w(\xi))\, dA(\xi) dr$$

This estimate will rely on size estimates. Since $0 < r < \frac{s}{2}$, $e^{-c\frac{|\xi-w|^2}{r}} < e^{-c\frac{|\xi-w|^2}{s}}$. Also,

$$e^{-c\frac{|\xi-w|}{s}} \leq e^{-c\frac{s}{\mu_p(w,1/\tau)^2}}, \qquad \text{if } s \leq |\xi-w|\mu_p(w,1/\tau). \tag{34}$$

Thus, we have (at most) two regions to consider: $\mu_p(w,1/\tau) \leq |\xi-w| \lesssim \frac{s}{\mu_p(w,1/\tau)}$ and $|\xi-w| \gtrsim \frac{s}{\mu_p(w,1/\tau)}$. The second region is not included in the first region when s is large (relative to $\mu_p(w,1/\tau)$). On the second region, by Lemma 6.5, Corollary 6.7, (34), and with a (possible) decrease of c, we have

$$\begin{aligned}
&\Big| \int_0^{\frac{s}{2}} \int_{|\xi-w| \gtrsim \frac{s}{\mu_p(w,1/\tau)}} H_{\tau p}(s-r,z,\xi) D^{K_1} e(w,\xi) Y^{K_2} Z_{\tau p,\xi} Y^{J-k-2} H_{\tau p}(r,\xi,w)(1-\varphi_w(\xi))\, dA(\xi) dr \Big| \\
&\lesssim \int_0^{\frac{s}{2}} \int_{\substack{|\xi-w| \geq \mu_p(w,1/\tau) \\ |\xi-w| \gtrsim \frac{s}{\mu_p(w,1/\tau)}}} \frac{e^{-c\frac{|\xi-z|^2}{s-r}}}{s-r} e^{-c\frac{s}{\mu_p(\xi,1/\tau)^2}} \frac{|D^{K_1} e(w,\xi)| \Lambda(\xi,\Delta)^{n-1}}{r^{\frac{1}{2}(1+\ell-|K_1|)}} \frac{e^{-c\frac{|\xi-w|^2}{r}}}{r} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}}\, dA(\xi) dr \\
&\lesssim \frac{\Lambda(z,\Delta)^n}{s^{1+\frac{\ell}{2}}} e^{-c\frac{|z-w|^2}{s}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} \int_0^{\frac{s}{2}} \int_{\mathbb{C}} \frac{e^{-c\frac{|\xi-z|^2}{s-r}}}{s-r} \frac{e^{-c\frac{|\xi-w|^2}{r}}}{r}\, dA(\xi) dr \lesssim \frac{\Lambda(z,\Delta)^n}{s^{1+\frac{\ell}{2}}} e^{-c\frac{|z-w|^2}{s}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}}.
\end{aligned}$$

The key size estimates for the region $\mu_p(w,1/\tau) \leq |\xi-w| \lesssim \frac{s}{\mu_p(w,1/\tau)}$ follow from the second inequality in Lemma 7.10 and $\frac{|\xi-w|}{\mu_p(w,1/\tau)} \lesssim \frac{s}{\mu_p(w,1/\tau)^2}$. Since $|\xi-w| \geq \mu_p(w,1/\tau)$, $\frac{\mu_p(w,1/\tau)}{\mu_p(\xi,1/\tau)} \gtrsim \big(\frac{\mu_p(w,1/\tau)}{|w-\xi|}\big)^{1-\delta}$, so with a

decrease in c,

$$
\begin{aligned}
e^{-c\left(\frac{s}{\mu_p(\xi,1/\tau)^2}\right)^{\epsilon}} = e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}(\frac{\mu_p(w,1/\tau)}{\mu_p(\xi,1/\tau)})^2\right)^{\epsilon}} &\gtrsim e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}(\frac{\mu_p(w,1/\tau)}{|\xi-w|})^{2-2\delta}\right)^{\epsilon}} \\
&\gtrsim e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}(\frac{s}{\mu_p(w,1/\tau)})^{2\delta-2}\right)^{\epsilon}} = e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^{2\delta\epsilon}} \qquad (35)
\end{aligned}
$$

With this estimate in hand, the size estimate follows from similar arguments as before. Indeed, with a decrease in c,

$$
\begin{aligned}
&\left|\int_0^{\frac{s}{2}}\int_{\mu_p(w,1/\tau)\le|\xi-w|\lesssim\frac{s}{\mu_p(w,1/\tau)}} H_{\tau p}(s-r,z,\xi)D^{K_1}e(w,\xi)Y^{K_2}Z_{\tau p,\xi}Y^{J-k-2}H_{\tau p}(r,\xi,w)(1-\varphi_w(\xi))\,dA(\xi)dr\right| \\
&\lesssim \int_0^{\frac{s}{2}}\int_{\mu_p(w,1/\tau)\le|\xi-w|\lesssim\frac{s}{\mu_p(w,1/\tau)}} \frac{e^{-c\frac{|\xi-z|^2}{s-r}}}{s-r} e^{-c\frac{s}{\mu_p(\xi,1/\tau)^2}} e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^{2\delta\epsilon}} \frac{|D^{K_1}e(w,\xi)|\Lambda(\xi,\Delta)^{n-1}}{r^{1+\frac{1}{2}(1+\ell-|K_1|)}} e^{-c\frac{|\xi-w|^2}{r}}\,dA(\xi)dr \\
&\lesssim \frac{\Lambda(z,\Delta)^n}{s^{1+\frac{\ell}{2}}} e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^{2\delta\epsilon}} \int_0^{\frac{s}{2}}\int_{\mathbb{C}} \frac{e^{-c\frac{|\xi-w|^2}{r}}}{r}\frac{e^{-c\frac{|\xi-z|^2}{s-r}}}{s-r}\,dA(\xi)dr \lesssim \frac{\Lambda(z,\Delta)^n}{s^{1+\frac{\ell}{2}}} e^{-c\frac{|z-w|^2}{s}} e^{-c\left(\frac{s}{\mu_p(w,1/\tau)^2}\right)^{2\delta\epsilon}}.
\end{aligned}
$$

Thus, in the case $Y_{J-k-2} = M_{\tau p}^{\xi,w}$, we have obtained the desired estimate. The remaining cases use no new ideas. If $Y_{J-k-2} = \overline{Z}_{\tau p,\xi}$ or $Z_{\tau p,\xi}$, we must integrate by parts (for the term with the $\overline{Z}_{\tau p,\xi}$ or $Z_{\tau p,\xi}$) as in Lemma 7.11 to put a $\overline{W}_{\tau p,\xi}$ or $W_{\tau p,\xi}$ on $H_{\tau p}(s-r,z,\xi)$. At which point, we can use the (n,γ)-size and cancellation conditions because the integration by parts guarantees that $\gamma < \ell$. The integral estimation is then similar to the one just performed. □

Remark 7.13. The curious ϵ in the definition of the (n,ℓ)-size condition is explained by (35).

7.5. **Proof of Theorem 4.10.**

Proof (Theorem 4.10). Proof by induction. The $(0,\infty)$-size and cancellation conditions are proved in [19, 21]. From Lemma 7.9, Lemma 7.11, and Proposition 7.12, it is clear that $H_{\tau p}(s,z,w)$ satisfies the (n,ℓ)-size and cancellation conditions for all n and ℓ. Thus, Theorem 4.10 is proved. □

7.6. **Proof of Theorem 4.2.** From the proof of Theorem 4.10, we know that $H_{\tau p}(s,z,w)$ satisfies the (n,ℓ)-size and cancellation conditions for all n and ℓ. Thus, the remaining estimate to show to finish the proof of Theorem 4.2 is the improved long time decay. We will use an integration by parts argument.

Recall a Sobolev embedding lemma from [21].

Theorem 7.14. *Let $\Delta = (a_1,b_1)\times(a_2,b_2) \subset \mathbb{R}^2$ be a square of sidelength δ. If $(x_1,x_2)\in\Delta$ and if $f\in\mathcal{C}^2(\Delta)$, then*

$$|f(x_1,x_2)|^2 \le 4\left(\frac{1}{\delta^2}\int_\Delta |f|^2 + \int_\Delta |\widetilde{\nabla} f|^2 + \delta^2\int_\Delta |X_2X_1f|^2\right).$$

Using the method of argument of the proof of Theorem 7.14, we show

Corollary 7.15. *Let $\Delta_1,\Delta_2\subset\mathbb{R}^2$ be squares of sidelength δ, and let $I = (\tau_0-\gamma,\tau_0+\gamma)\subset\mathbb{R}$. If $(z,w,\tau)\in\Delta_1\times\Delta_2\times I$ and if $f\in\mathcal{C}^4(\Delta_1\times\Delta_2\times I)$, then*

$$|f(z,w,\tau)|^2 \le \frac{64}{\delta^4\gamma}\sum_{K=(k,j)\le(1,4)} \delta^{2j}\gamma^{2k}\|Y^Kf\|^2_{L^2(\Delta_1\times\Delta_2\times I)}.$$

Proof. For $h\in\mathcal{C}^1(\mathbb{R})$, $h(\tau) = h(x) + \int_x^\tau h'(\sigma)\,d\sigma$. Integrating in x over I, we have

$$2\gamma|h(\tau)| \le \int_I |h(\sigma)|\,d\sigma + 2\gamma\int_I |h'(\sigma)|\,d\sigma.$$

Applying Cauchy-Schwarz and squaring, we have

$$|h(\tau)|^2 \le 4\bigg(\frac{1}{\gamma}\int_I |h(\sigma)|^2\,d\sigma + \gamma\int_I |h'(\sigma)|^2\,d\sigma\bigg).$$

$T(w,z)$ is $\mathbb{R}$-valued, so if $g(\sigma) = e^{-i\sigma T\left(w,(x_1,x_2)\right)}h(\sigma)$, then

$$|g(\tau)|^2 \le 4\bigg(\frac{1}{\gamma}\int_I |g(\sigma)|^2\,d\sigma + \gamma\int_I |M_{\sigma p}g(\sigma)|^2\,d\sigma\bigg). \tag{36}$$

To finish the proof, we apply Theorem 7.14 to f twice: once in z and once in w (with U_1 and U_2 replacing X_1 and X_2) for each term from the z estimate. This gives

$$|f(z,w,\tau)|^2 \le \frac{16}{\delta^4}\sum_{K=(0,j),j\le 4}\delta^{2|K|}\|Y^K f(\cdot,\cdot,\tau)\|^2_{L^2(\Delta_1\times\Delta_2)}.$$

Applying (36) to each term in the previous inequality finishes the proof. □

Lemma 7.16. *Let $z,w\in\mathbb{C}$ and $\tau\in\mathbb{R}$ and $J\in(n,\ell)$. If $\delta,\gamma>0$, $B = D(z,\delta)\times D(w,\delta)\times(\tau-\gamma,\tau+\gamma)$, and $F\in\mathcal{C}^\infty(B)$, then there exits $C>0$ so that*

$$|Y^J F(z,w,\tau)|^2 \le C\max_{K\in(k,j)\le(2n+2,2\ell+8)}\delta^{j-2\ell}\gamma^{k-2n}\|Y^K F\|_{L^\infty(B)}\|F\|_{L^\infty(B)}.$$

Proof. Let $\psi\in\mathcal{C}^\infty_c(\mathbb{C}\times\mathbb{C}\times\mathbb{R})$ so that $\operatorname{supp}\psi\subset B$, $\psi(z,w,\tau)=1$, and $|\frac{\partial^k}{\partial\tau^k}\nabla^j\psi|\le C_{j,k}\delta^{-j}\gamma^{-k}$. By Corollary 7.15,

$$|Y^J F(z,w,\tau)|^2 = |Y^J F(z,w,\tau)\psi(z,w,\tau)|^2 \lesssim \frac{1}{\delta^4\gamma}\sum_{L=(k,j)\le(1,4)}\delta^{2j}\gamma^{2k}\big\|Y^L\big(Y^J F\psi\big)\big\|_{L^2(B)}.$$

Since

$$\big\|Y^L\big(Y^J F\psi\big)\big\|_{L^2(B)} \lesssim \sum_{\substack{J_1\in(n+k_1,\ell+j_1)\\ 0\le j_1\le j\\ 0\le k_1\le k}}\Big\|(Y^{J_1}F)\,\nabla^{j-j_1}\frac{\partial^{k-k_1}}{\partial\tau^{k-k_1}}\psi\Big\|_{L^2(B)}.$$

Let χ_B be the characteristic function of B. Taking an arbitrary term from the sum, we have

$$\begin{aligned}
&\Big\|(Y^{J_1}F)\,\nabla^{j-j_1}\frac{\partial^{k-k_1}}{\partial\tau^{k-k_1}}\psi\Big\|^2_{L^2(B)} = \Big((Y^{J_1}F)\,\nabla^{j-j_1}\frac{\partial^{k-k_1}}{\partial\tau^{k-k_1}}\psi, (Y^{J_1}F)\,\nabla^{j-j_1}\frac{\partial^{k-k_1}}{\partial\tau^{k-k_1}}\psi\Big)\\
&= \bigg|\bigg(Y^{J_1}\Big[(Y^{J_1}F)\,\Big(\nabla^{j-j_1}\frac{\partial^{k-k_1}}{\partial\tau^{k-k_1}}\psi\Big)\Big(\nabla^{j-j_1}\frac{\partial^{k-k_1}}{\partial\tau^{k-k_1}}\psi\Big)\Big], F\chi_B\bigg)\bigg|\\
&\lesssim \sum_{\substack{K\in(n+k_1+k_1^1,\ell+j_1+j_1^1)\\ j_1^1+j_1^2+j_1^3=\ell+j_1\\ k_1^1+k_1^2+k_1^3=n+k_1}}\bigg|\bigg((Y^K F)\Big(\nabla^{j-j_1+j_1^2}\frac{\partial^{k-k_1}}{\partial\tau^{k-k_1+k_1^2}}\psi\Big)\Big(\nabla^{j-j_1+j_1^3}\frac{\partial^{k-k_1}}{\partial\tau^{k-k_1+k_1^3}}\psi\Big), F\chi_B\bigg)\bigg|.
\end{aligned}$$

Thus, it is enough to estimate

$$\begin{aligned}
&\frac{1}{\delta^4\gamma}\delta^{2j}\gamma^{2k}\bigg|\bigg((Y^K F)\Big(\nabla^{j-j_1+j_1^2}\frac{\partial^{k-k_1}}{\partial\tau^{k-k_1+k_1^2}}\psi\Big)\Big(\nabla^{j-j_1+j_1^3}\frac{\partial^{k-k_1}}{\partial\tau^{k-k_1+k_1^3}}\psi\Big), F\chi_B\bigg)\bigg|\\
&\qquad\lesssim \delta^{2j}\tau^{2k}\frac{1}{\delta^{j-j_1+j_1^2}\gamma^{k-k_1+k_1^2}}\frac{1}{\delta^{j-j_1+j_1^3}\gamma^{k-k_1+k_1^3}}\|Y^K F\|_{L^\infty(B)}\|F\|_{L^\infty(B)}.
\end{aligned}$$

However, $\delta^{-2j+j-j_1+j_1^2+j-j_1+j_1^3} = \delta^{-2j_1+j_1^2+j_1^3} = \delta^{\ell-j_1-j_1^1}$ since $j_1^1+j_1^2+j_1^3 = \ell+j_1$. Similarly, since $k_1^1+k_1^2+k_1^3 = n+k_1$, $\gamma^{-2n+k-k_1+k_1^2+k-k_1+k_1^3} = \gamma^{n-k_1-k_1^1}$. Thus,

$$|Y^J F(z,w,\tau)|^2 \lesssim \max_{\substack{K\in(n+k_1+k_1^1,\ell+j_1+j_1^1)\\ 0\le j_1\le 4\\ 0\le j_1^1\le\ell+4}}\frac{1}{\delta^{\ell-j_1-j_1^1}\gamma^{n-k_1-k_1^1}}\|Y^K F\|_{L^\infty(B)}\|F\|_{L^\infty(B)}.$$

□

We will use Lemma 7.16 to prove Theorem 4.2.

Proof (Theorem 4.2). We now recover the superior estimates of Theorem 4.2. Fix $(s, z, w) \in (0, \infty) \times \mathbb{C} \times \mathbb{C}$. Since $H_{\tau p}(s, z, w)$ satisfies (1), it is enough to estimate $Y^J H_{\tau p}(s, z, w)$ where $J \in (n, \ell)$ since s-derivatives can be written in terms of $\Box_{\tau p}$.

We already have the estimate

$$|Y^\alpha H_{\tau p}(s, z, w)| \lesssim \frac{\Lambda(z, s^{1/2})^n}{s^{1+\frac{\ell}{2}}} e^{-c\frac{|z-w|^2}{s}}. \tag{37}$$

Since $\frac{1}{\tau} > \Lambda(z, s^{1/2})$ means $\mu_p(z, 1/\tau) > s^{1/2}$, (37) is the estimate in the τ-small case. Thus, we have left to show

$$|Y^\alpha H_{\tau p}(s, z, w)| \lesssim \frac{1}{\tau^n s^{1+\frac{\ell}{2}}} e^{-c\frac{|z-w|^2}{s}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}}.$$

Let $\delta = \frac{1}{2}\Delta$ and $\gamma = \frac{1}{4}\tau$. Since $\mu_p(z, r) \sim \mu_p(z, 2r)$ for all z with a constant depending only on $\deg p$, it follows that $\mu_p(\xi, \sigma) \sim \mu_p(z, 1/\tau)$ for all $(\xi, \sigma) \in D(z, \delta) \times (\tau - \gamma, \tau + \gamma)$. Thus, using the notation of Lemma 7.16, we have $F(z, w, \tau) = H_{\tau p}(s, z, w)$, and $\|Y^K H_{\tau p}\|_{L^\infty(B)} \sim |Y^K H_{\tau p}(s, z, w)|$. We bound $|H_{\tau p}(s, z, w)|$ from the known $(0, \ell)$-estimate of Theorem 4.2 from [19] and $|Y^K H_{\tau p}(s, z, w)|$ from Proposition 7.12. Since $\delta \le s^{1/2}$, with a decrease in c,

$$\begin{aligned}
|Y^J H_{\tau p}(s, z, w)| &\lesssim \max_{K \in (k,j) \le (2n+2, 2\ell+8)} \delta^{\frac{j}{2}-\ell} \tau^{\frac{k}{2}-n} |Y^K H_{\tau p}(s, z, w)|^{\frac{1}{2}} |H_{\tau p}(s, z, w)|^{\frac{1}{2}} \\
&\lesssim \max_{K \in (k,j) \le (2n+2, 2\ell+8)} \delta^{\frac{j}{2}-\ell} \tau^{\frac{k}{2}-n} \frac{1}{\tau^{k/2} s^{1+j/2}} e^{-c\frac{|z-w|^2}{s}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}} \\
&\lesssim \frac{1}{\tau^n s^{1+\frac{1}{2}\ell}} e^{-c\frac{|z-w|^2}{s}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} e^{-c\frac{s}{\mu_p(z,1/\tau)^2}}.
\end{aligned}$$

$\square$

8. Size estimates for $Y^\alpha \tilde{G}_{\tau p}(s, z, w)$

We now turn to the proof of Theorem 4.5.

Proof. (Theorem 4.5). Since $\tilde{G}_{\tau p}(s, z, w) = \overline{\tilde{G}_{\tau p}(s, w, z)}$, it is enough to show

$$|Y^\alpha \tilde{G}_{\tau p}(s, z, w)| \le \frac{C_{|\alpha|}}{\tau^n} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} \max\Big\{ \frac{e^{-c\frac{|z-w|^2}{s}}}{s^{1+\frac{1}{2}\ell}}, \frac{e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}}}{\mu_p(z, 1/\tau)^{2+\ell}} \Big\}.$$

From Proposition 4.7,

$$Y^\alpha \tilde{G}_{\tau p}(s, z, w) = -\int_{\mathbb{C}} Y^\alpha \Big[\overline{W}_{\tau p, w} H_{\tau p}(s, v, w) R_{\tau p}(z, v)\Big]\, dA(v).$$

We expand $Y^\alpha \Big[\overline{W}_{\tau p, w} H_{\tau p}(s, v, w) R_{\tau p}(z, v)\Big]$ and use Proposition 7.5 to see

$$Y^\alpha \Big[\overline{W}_{\tau p, w} H_{\tau p}(s, v, w) R_{\tau p}(z, v)\Big] = \sum_{\substack{n_1+n_2+n_3=n \\ \ell_1+\ell_2+\sum_{j=1}^{n_3}|\beta_j| = \ell \\ \alpha_1 \in (n_1,\ell_1), \alpha_2 \in (n_2,\ell_2)}} c_{\alpha_1,\alpha_2,\beta_j} Y^{\alpha_1} \overline{W}_{\tau p, w} H_{\tau p}(s, v, w) Y^{\alpha_2} R_{\tau p}(z, v) \Big(\prod_{j=1}^{n_3} D^{\beta_j}_{w,z} r(w, v, z) \Big).$$

It is enough to take one term from the sum and estimate its integral over $\mathbb{C}$ (in v).

First, assume that $|z - w| \ge 2\mu_p(w, 1/\tau)$ and $|z - w| \ge 2\mu_p(z, 1/\tau)$. We decompose the integral into four pieces: near w, near z, and away from z and w (will be two integrals). For $x \in \mathbb{C}$, let $\varphi_x \in C^\infty(\mathbb{C})$ so that $\varphi_x \equiv 1$ on $D\big(x, \mu_p(x, 1/\tau)/2\big)$, $\operatorname{supp} \varphi_x \subset D(x, \mu_p(x, 1/\tau))$, $0 \le \varphi_x \le 1$, and $|D^\alpha \varphi_x| \lesssim |\mu_p(x, 1/\tau)|^{-|\alpha|}$ with constants independent of x. Note that $\operatorname{supp} \varphi_z \cap \operatorname{supp} \varphi_w = \emptyset$.

Near z, we use the $R_{\tau p}$-cancellation conditions from Corollary 4.11. Our first estimate is:

$$\begin{aligned}
&\left|\int_{\mathbb{C}} Y^{\alpha_1}\overline{W}_{\tau p,w}H_{\tau p}(s,v,w)Y^{\alpha_2}R_{\tau p}(z,v)\Big(\prod_{j=1}^{n_3}D^{\beta_j}_{w,z}r(w,v,z)\Big)\varphi_z(v)\big(1-\varphi_w(v)\big)\,dA(v)\right| \\
&=\left|\int_{\mathbb{C}} Y^{\alpha_1}\overline{W}_{\tau p,w}H_{\tau p}(s,v,w)Y^{\alpha_2}R_{\tau p}(z,v)\Big(\prod_{j=1}^{n_3}D^{\beta_j}_{w,z}r(w,v,z)\Big)\varphi_z(v)\,dA(v)\right| \\
&=\left|Y^{\alpha_2}R_{\tau p}\Big[Y^{\alpha_1}\overline{W}_{\tau p,w}H_{\tau p}(s,\cdot,w)\Big(\prod_{j=1}^{n_3}D^{\beta_j}_{w,z}r(w,\cdot,z)\Big)\varphi_z\Big](z)\right| \qquad (38)
\end{aligned}$$

The cases ℓ_2 even and ℓ_2 odd are estimated similarly, and we will only show the $\ell_2=2k-1$ odd case. Recall that $R_{\tau p}=Z_{\tau p}G_{\tau p}$, so by Corollary 4.11, we can estimate (38) by

$$\begin{aligned}
&\frac{C_{|\alpha_2|}}{\tau^{n_2}}\Big(\delta\Big\|\Box^k_{\tau p,v}\Big(Y^{\alpha_1}\overline{W}_{\tau p,w}H_{\tau p}(s,\cdot,w)\big(\prod_{j=1}^{n_3}D^{\beta_j}_{w,z}r(w,\cdot,z)\big)\varphi_z\Big)\Big\|_{L^2(\mathbb{C})} \\
&\qquad\qquad+\delta^3\Big\|\Box^{k+1}_{\tau p,v}\Big(Y^{\alpha_1}\overline{W}_{\tau p,w}H_{\tau p}(s,\cdot,w)\big(\prod_{j=1}^{n_3}D^{\beta_j}_{w,z}r(w,\cdot,z)\big)\varphi_z\Big)\Big\|_{L^2(\mathbb{C})}\Big). \qquad (39)
\end{aligned}$$

The two terms in (39) are estimated in the same fashion, and we only present the estimate of the first term.

$$\begin{aligned}
&\Box^k_{\tau p,v}\Big(Y^{\alpha_1}\overline{W}_{\tau p,w}H_{\tau p}(s,v,w)\big(\prod_{j=1}^{n_3}D^{\beta_j}_{w,z}r(w,v,z)\big)\varphi_z(v)\Big) \\
&=\sum_{|\gamma_0|+\cdots+|\gamma_{n_3+1}|=2k} c_{\gamma_j}\,X^{\gamma_0}Y^{\alpha_1}\overline{W}_{\tau p,w}H_{\tau p}(s,v,w)\big(\prod_{j=1}^{n_3}D^{\gamma_j}_v D^{\beta_j}_{w,z}r(w,v,z)\big)D^{\gamma_{n_3+1}}\varphi_z(v) \qquad (40)
\end{aligned}$$

It is enough to estimate the L^2 norm of an arbitrary term in the expansion of (40). On $\operatorname{supp}\varphi_z$, note that $|w-v|\sim|z-w|$ and $\mu_p(v,1/\tau)\sim\mu_p(z,1/\tau)$. Also, since $|v-w|\sim|z-w|\geq\mu_p(z,1/\tau)$, we can interchange s with $\mu_p(z,1/\tau)$ at will because of our exponential factors (though we may have to decrease c with each substitution). Then

$$\begin{aligned}
&\frac{\mu_p(z,1/\tau)}{\tau^{n_2}}\Big\|X^{\gamma_0}Y^{\alpha_1}\overline{W}_{\tau p,w}H_{\tau p}(s,\cdot,w)\Big(\prod_{j=1}^{n_3}D^{\gamma_j}_v D^{\beta_j}_{w,z}r(w,\cdot,z)\Big)D^{\gamma_{n_3+1}}\varphi_z\Big\|_{L^2(\mathbb{C})} \\
&\lesssim\frac{\mu_p(z,1/\tau)}{\tau^{n_2}}\Big(\int_{\operatorname{supp}\varphi_z}\tau^{-2n_1}\frac{e^{-c\frac{|v-w|^2}{s}}e^{-c\frac{s}{\mu_p(v,1/\tau)^2}}e^{-c\frac{s}{\mu_p(w,1/\tau)^2}}}{s^{2+|\gamma_0|+\ell_1+1}}\prod_{j=1}^{n_3}\big|D^{\gamma_j}_vD^{\beta_j}_{w,z}r(w,v,z)\big|^2\frac{1}{\mu_p(z,1/\tau)^{2\gamma_{n_3+1}}}\,dv\Big)^{1/2} \\
&\lesssim\frac{s}{\tau^n}\frac{1}{s^{1+\frac12(|\gamma_0|+\cdots+|\gamma_{n_3+1}|+\ell_1+|\beta_1|+\cdots+|\beta_{n_3}|+1)}}e^{-c\frac{|z-w|^2}{s}}e^{-c\frac{s}{\mu_p(z,1/\tau)^2}}e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} \\
&=\frac{1}{\tau^n s^{1+\frac12\ell}}e^{-c\frac{|z-w|^2}{s}}e^{-c\frac{s}{\mu_p(z,1/\tau)^2}}e^{-c\frac{s}{\mu_p(w,1/\tau)^2}},
\end{aligned}$$

since $\ell_1+\sum|\beta_j|=\ell-\ell_2$ and $|\gamma_0|+\cdots+|\gamma_{n_3+1}|=2k=\ell_2+1$. This is (better than) the desired estimate.

We begin the estimate for region near w. We first find the estimate for the case $s^{1/2}\leq\mu_p(w,1/\tau)$. We can assume that ℓ_1 is odd because the ℓ_1 even case is handled analogously. If we set $\delta=\mu_p(w,1/\tau)$, by

Theorem 4.10 we estimate:

$$\begin{aligned}
&\Big|\int_{\mathbb{C}} Y^{\alpha_1}\overline{W}_{\tau p,w}H_{\tau p}(s,v,w)Y^{\alpha_2}R_{\tau p}(z,v)\Big(\prod_{j=1}^{n_3}D^{\beta_j}_{w,z}r(w,v,z)\Big)\varphi_w(v)\,dA(v)\Big| \\
&\qquad\qquad = \Big|Y^{\alpha_1}\overline{W}_{\tau p,w}(H^s_{\tau p})^{\#}\Big[Y^{\alpha_2}R_{\tau p}(z,\cdot)\Big(\prod_{j=1}^{n_3}D^{\beta_j}_{w,z}r(w,\cdot,z)\Big)\varphi\Big](w)\Big| \\
&\qquad\qquad \lesssim \frac{1}{\tau^{n_1}\delta}\Big(\Big\|(\square^{\#}_{\tau p})^{\frac{\ell_1+1}{2}}\Big(Y^{\alpha_2}R_{\tau p}(z,\cdot)\big(\prod_{j=1}^{n_3}D^{\beta_j}_{w,z}r(w,\cdot,z)\big)\varphi_w\Big)\Big\|_{L^2} \\
&\qquad\qquad + \delta^2\Big\|(\square^{\#}_{\tau p})^{\frac{\ell_1+3}{2}}\Big(Y^{\alpha_2}R_{\tau p}(z,\cdot)\big(\prod_{j=1}^{n_3}D^{\beta_j}_{w,z}r(w,\cdot,z)\big)\varphi_w\Big)\Big\|_{L^2}\Big).
\end{aligned}$$

The two terms are handled similarly. We estimate the first term.

$$\begin{aligned}
&\frac{1}{\tau^{n_1}\delta}\Big\|(\square^{\#}_{\tau p})^{\frac{\ell_1+1}{2}}\Big(Y^{\alpha_2}R_{\tau p}(z,\cdot)\big(\prod_{j=1}^{n_3}D^{\beta_j}_{w,z}r(w,\cdot,z)\big)\varphi_w\Big)\Big\|_{L^2(\mathbb{C})} \\
&\qquad \lesssim \frac{1}{\tau^{n_1}\delta}\sum_{|\gamma_0|+\cdots+|\gamma_{n_3}+1|=\ell_1+1}\big\|X^{\gamma_0}Y^{\alpha_2}R_{\tau p}(z,\cdot)\big(\prod_{j=1}^{n_3}D^{\beta_j}_{w,z}r(w,\cdot,z)\big)D^{\gamma_{n_3}+1}\varphi_w\big\|_{L^2(\mathbb{C})}.
\end{aligned}$$

We pick an arbitrary term from the sum to bound. On $\operatorname{supp}\varphi_w$, note that $|v-z|\sim|w-z|$ and $\mu_p(v,1/\tau)\sim\mu_p(w,1/\tau)$.

$$\begin{aligned}
&\big\|X^{\gamma_0}Y^{\alpha_2}R_{\tau p}(z,\cdot)\big(\prod_{j=1}^{n_3}D^{\beta_j}_{w,z}r(w,\cdot,z)\big)D^{\gamma_{n_3}+1}\varphi_w\big\|_{L^2(\mathbb{C})} \\
&\frac{1}{\tau^{n_1}\delta}\Big(\int_{\operatorname{supp}\varphi_w}\frac{1}{\mu_p(z,1/\tau)^{2+2|\gamma_0|+2\ell_2}\tau^{2n_2}}e^{-c\frac{|z-v|}{\mu_p(v,1/\tau)}}e^{-c\frac{|z-v|}{\mu_p(z,1/\tau)}}\prod_{j=1}^{n_3}|D^{\beta_j}_{w,z}r(w,v,z)|^2\frac{1}{\delta^{2\gamma_{n_3+1}}}\,dv\Big)^{1/2} \\
&\lesssim \frac{1}{\tau^n}\frac{1}{\mu_p(z,1/\tau)^{1+\ell_2+|\gamma_0|+\cdots+|\gamma_{n_3+1}|+|\beta_1|+\cdots+|\beta_{n_3}|}}e^{-c\frac{|z-w|}{\mu_p(w,1/\tau)}}e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}} \\
&= \frac{1}{\tau^n\mu_p(z,1/\tau)^{2+\ell}}e^{-c\frac{|z-w|}{\mu_p(w,1/\tau)}}e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}}
\end{aligned}$$

since $|\gamma_0|+\cdots+|\gamma_{n_3+1}|=\ell_1+1$ and $\ell_1+\ell_2+\sum|\beta_j|=\ell$. This is the desired estimate in the case $s^{\frac12}\le\mu_p(w,1/\tau)$. If $s^{\frac12}\ge\mu_p(w,1/\tau)$, our estimate follows from size estimates. Indeed,

$$\begin{aligned}
&\Big|\int_{\mathbb{C}} Y^{\alpha_1}\overline{W}_{\tau p,w}H_{\tau p}(s,v,w)Y^{\alpha_2}R_{\tau p}(z,v)\Big(\prod_{j=1}^{n_3}D^{\beta_j}r(w,v,z)\Big)\varphi_w(v)\,dA(v)\Big| \\
&\lesssim \frac{\mu_p(w,1/\tau)^2}{\tau^{n_1}s^{1+\frac12\ell_1+\frac12}}e^{-c\frac{s}{\mu_p(w,1/\tau)^2}}e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}}e^{-c\frac{|z-w|}{\mu_p(w,1/\tau)}}\frac{1}{\tau^{n_2}\mu_p(w,1/\tau)^{1+\ell_2}}\frac{1}{\tau^{n_3}\mu_p(w,1/\tau)^{\sum|\beta_j|}} \\
&= \frac{1}{\tau^n s^{1+\frac12\ell}}e^{-c\frac{s}{\mu_p(w,1/\tau)^2}}e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}}e^{-c\frac{|z-w|}{\mu_p(w,1/\tau)}}. \qquad (41)
\end{aligned}$$

The remaining two estimates simply use the size conditions from Theorem 4.2 and Corollary 4.8. The third integral we estimate is on the region $|v-w| \geq |v-z|$. On this region, $|v-w| \geq \frac{1}{2}|z-w|$, so

$$
\begin{aligned}
&\Big| \int_{|v-w|\geq|v-z|} Y^{\alpha_1}\overline{W}_{\tau p,w} H_{\tau p}(s,v,w) Y^{\alpha_2} R_{\tau p}(z,v) \Big(\prod_{j=1}^{n_3} D^{\beta_j}_{w,z} r(w,v,z)\Big)\big(1-\varphi_z(v)\big)\big(1-\varphi_w(v)\big)\, dA(v)\Big| \\
&\lesssim \int_{|v-w|\geq|v-z|} \frac{1}{\tau^{n_1} s^{1+\frac{1}{2}(\ell_1+1)}} e^{-c\frac{|v-w|^2}{s}} e^{-c\frac{s}{\mu_p(v,1/\tau)^2}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} \frac{1}{\mu_p(v,1/\tau)^{1+\ell_2}} \Big| \prod_{j=1}^{n_3} D^{\beta_j}_{w,z} r(w,v,z)\Big|\, dA(v) \\
&\lesssim \frac{1}{\tau^n} \frac{1}{s^{1+\frac{1}{2}(\ell_1+\ell_2)}} \frac{1}{s^{\frac{1}{2}(|\beta_1|+\cdots|\beta_{n_3}|)}} e^{-c\frac{|z-w|^2}{s}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} \int_{\mathbb{C}} \frac{1}{s} e^{-c\frac{|v-w|^2}{s}}\, dA(v) \\
&\lesssim \frac{1}{\tau^n s^{1+\frac{1}{2}\ell}} e^{-c\frac{|z-w|^2}{s}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}}.
\end{aligned}
$$

The final integral is over the region $|v-w| \leq |v-z|$. In this case, $|v-z| \geq \frac{1}{2}|w-z|$. We estimate

$$
\begin{aligned}
&\Big| \int_{|v-w|\leq|v-z|} Y^{\alpha_1}\overline{W}_{\tau p,w} H_{\tau p}(s,v,w) Y^{\alpha_2} R_{\tau p}(z,v) \Big(\prod_{j=1}^{n_3} D^{\beta_j}_{w,z} r(w,v,z)\Big)\big(1-\varphi_z(v)\big)\big(1-\varphi_w(v)\big)\, dA(v)\Big| \\
&\lesssim \int_{|v-w|\leq|v-z|} \frac{e^{-c\frac{|v-w|^2}{s}}}{\tau^{n_1} s^{1+\frac{1}{2}(\ell_1+1)}} e^{-c\frac{s}{\mu_p(v,1/\tau)^2}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} \frac{e^{-c\frac{|z-v|}{\mu_p(v,1/\tau)}} e^{-c\frac{|z-v|}{\mu_p(z,1/\tau)}}}{\mu_p(v,1/\tau)^{1+\ell_2}} \Big| \prod_{j=1}^{n_3} D^{\beta_j}_{w,z} r(w,v,z)\Big|\, dA(v) \\
&\lesssim \frac{1}{\tau^n} \frac{1}{\mu_p(z,1/\tau)^{\ell_1+\ell_2+1+\sum|\beta_j|}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}} \int_{\mathbb{C}} \frac{1}{s} e^{-c\frac{|v-w|^2}{s}}\, dA(v) \\
&\lesssim \frac{1}{\tau^n \mu_p(z,1/\tau)^{2+\ell}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}}.
\end{aligned}
$$

We have completed the estimates for the case $|z-w| \geq 2\mu_p(z,1/\tau)$ and $2\mu_p(w,1/\tau)$.

The cases $|z-w| \leq 2\mu_p(z,1/\tau)$ and $|z-w| \leq 2\mu_p(w,1/\tau)$ are similar to the estimates already performed. An important feature of the near-diagonal estimate is that $\mu_p(z,1/\tau) \sim \mu_p(w,1/\tau)$. Using size estimates and mimicking the techniques used earlier in this proof, with a decrease in c (to help turn s into $\mu_p(w,1/\tau)$), we can show

$$
\begin{aligned}
\Big| \int_{\mathbb{C}} Y^{\alpha_1}\overline{W}_{\tau p,w} H_{\tau p}(s,v,w) Y^{\alpha_2} R_{\tau p}(z,v) \Big(\prod_{j=1}^{n_3} D^{\beta_j}_{w,z} r(w,v,z)\Big)\big(1-\varphi_z(v)\big)\big(1-\varphi_w(v)\big)\, dA(v)\Big| \\
\lesssim \frac{1}{\mu_p(w,1/\tau)^{2+\ell}} e^{-c\frac{s}{\mu_p(w,1/\tau)^2}} e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}}.
\end{aligned}
$$

The estimation of the near w integral

$$
\begin{aligned}
&\int_{\mathbb{C}} Y^{\alpha_1}\overline{W}_{\tau p,w} H_{\tau p}(s,v,w) Y^{\alpha_2} R_{\tau p}(z,v) \Big(\prod_{j=1}^{n_3} D^{\beta_j}_{w,z} r(w,v,z)\Big)\big(1-\varphi_z(v)\big)\varphi_w(v)\, dA(v) \\
&\qquad = Y^{\alpha_1}\overline{W}_{\tau p,w}(H^s_{\tau p})^{\#}\Big[R_{\tau p}(z,\cdot)\Big(\prod_{j=1}^{n_3} D^{\beta_j}_{w,z} r(w,\cdot,z)\Big)\big(1-\varphi_z(\cdot)\big)\varphi_w(\cdot)\Big](w),
\end{aligned}
$$

proceeds as before with the $H^s_{\tau p}$-cancellation conditions and Theorem 4.10. Also, in the case that $s^{\frac{1}{2}} \geq \mu_p(w,1/\tau)$, the integral estimate (41) suffices in the near-diagonal case. Finally, the near z integral

$$
\begin{aligned}
&\int_{\mathbb{C}} Y^{\alpha_1}\overline{W}_{\tau p,w} H_{\tau p}(s,v,w) Y^{\alpha_2} R_{\tau p}(z,v) \Big(\prod_{j=1}^{n_3} D^{\beta_j}_{w,z} r(w,v,z)\Big)\big(1-\varphi_z(v)\big)\varphi_w(v)\, dA(v) \\
&\qquad = Y^{\alpha_2} R_{\tau p}\Big[Y^{\alpha_1}\overline{W}_{\tau p,w} H_{\tau p}(s,\cdot,w)\Big(\prod_{j=1}^{n_3} D^{\beta_j}_{w,z} r(w,\cdot,z)\Big)\big(1-\varphi_w(\cdot)\big)\varphi_z(\cdot)\Big](z),
\end{aligned}
$$

and the estimate follows from the $R_{\tau p}$-cancellation condition, Corollary 4.11. The proof of theorem is complete with the observation that

$$\min_{s\geq 0} \frac{|z-w|^2}{s} + \frac{s}{\mu_p(z,1/\tau)^2} = \frac{|z-w|}{\mu_p(z,1/\tau)}. \tag{42}$$

which allows to pull the $e^{-c\frac{|z-w}{\mu_p(z,1/\tau)}}e^{-c\frac{|z-w}{\mu_p(w,1/\tau)}}$ out of the max. $\square$

9. Size Estimates of $Y^J\tilde{H}_{\tau p}(s,z,w)$ – Proof of Theorem 4.4

The estimation of $Y^J\tilde{H}_{\tau p}(s,z,w)$ follows from ideas we have already used and Theorem 4.5.

Proof. (Theorem 4.4). We know $Y^J\tilde{H}_{\tau p}(s,z,w) = Y^J\tilde{G}_{\tau p}(s,z,w) + Y^J S_{\tau p}(z,w)$. From Theorem 4.5 and Corollary 4.11 and (42), we have the bound

$$|Y^J\tilde{H}_{\tau p}(s,z,w)| \lesssim \frac{1}{\tau^n}\max\left\{\frac{e^{-c\frac{|z-w|^2}{s}}e^{-c\frac{s}{\mu_p(w,1/\tau)^2}}e^{-c\frac{s}{\mu_p(z,1/\tau)^2}}}{s^{1+\frac{1}{2}\ell}}, \frac{e^{-c\frac{|z-w|}{\mu_p(w,1/\tau)}}e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}}}{\mu_p(w,1/\tau)^{2+\ell}}\right\} \tag{43}$$

$$\leq \frac{1}{\tau^n}e^{-c\frac{|z-w|}{\mu_p(w,1/\tau)}}e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}}\max\left\{\frac{1}{s^{1+\frac{1}{2}\ell}}, \frac{1}{\mu_p(w,1/\tau)^{2+\ell}}\right\}. \tag{44}$$

We will move the Gaussian decay term outside of the brackets using Lemma 7.16.

Fix (s,z,w) and $\tau>0$. As in the proof of Theorem 4.2, we let $\delta = \frac{1}{2}\min\{s^{\frac{1}{2}}, \mu_p(w,1/\tau), \mu_p(z,1/\tau)\}$ and $\gamma = \frac{1}{4}\tau$. Set $F(z,w,\tau) = \tilde{H}_{\tau p}(s,z,w)$. Then $\|Y^K\tilde{H}_{\tau p}\|_{L^\infty(B)} \sim |Y^K\tilde{H}_{\tau p}(s,z,w)|$. We use the bound for $|Y^K\tilde{H}_{\tau p}(s,z,w)|$ from (44) and the bound for $|\tilde{H}_{\tau p}(s,z,w)|$ from the known $(0,\ell)$-case of Theorem 4.4. If $\delta = \frac{1}{2}\mu_p(w,1/\tau)$, then

$$\begin{aligned}
|Y^J\tilde{H}_{\tau p}(s,z,w)| &\lesssim \max_{K\in(k,j)\leq(2n+2,2\ell+8)} \delta^{\frac{j}{2}-\ell}\tau^{\frac{k}{2}-n}|Y^K\tilde{H}_{\tau p}(s,z,w)|^{\frac{1}{2}}|\tilde{H}_{\tau p}(s,z,w)|^{\frac{1}{2}}\\
&\lesssim \max_{K\in(k,j)\leq(2n+2,2\ell+8)} \frac{1}{\tau^{k/2}}\delta^{\frac{j}{2}-\ell}\tau^{\frac{k}{2}-n}e^{-c\frac{|z-w|}{\mu_p(w,1/\tau)}}e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}}\max\left\{\frac{1}{s^{\frac{1}{2}+\frac{j}{4}}}, \frac{1}{\mu_p(w,1/\tau)^{1+j/2}}\right\}\\
&\times e^{-c\frac{|z-w|^2}{s}}\max\left\{\frac{1}{s^{\frac{1}{2}}}, \frac{1}{\mu_p(w,1/\tau)}\right\}\\
&\lesssim \frac{1}{\tau^n}e^{-c\frac{|z-w|^2}{s}}e^{-c\frac{|z-w|}{\mu_p(w,1/\tau)}}e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}}\max\left\{\frac{1}{s^{1+\frac{\ell}{2}}}, \frac{1}{\mu_p(w,1/\tau)^{2+\ell}}\right\}.
\end{aligned}$$

This is the desired estimate in the case $\mu_p(z,1/\tau)\geq s^{1/2}$.

The final case is when $s^{1/2}\leq\mu_p(w,1/\tau)$. Consider the following: we know the result holds if $Y^J\in(0,\infty)$. Assume the result holds if $Y^J\in(n-1,\infty)$. Let $\varphi\in\mathcal{C}_c^\infty(B(z,\delta))$ so that $\varphi\equiv 1$ on $B(z,\delta/2)$ and $|\nabla^k\varphi|\leq c_k/\delta^k$ for $k\leq 3$. Choose $\delta<\Delta$ small enough so that $|Y^K H_{\tau p}(s,\xi,w)|\sim|Y^K H_{\tau p}(s,z,w)|$ if $Y^K = X^\alpha Y^J$ and $|\alpha|\leq 2$. By argument leading up to (23), it follows that

$$M^{z,w}_{\tau p}Y^J\tilde{H}_{\tau p}(s,z,w) = M^{z,w}_{\tau p}\big(Y^J\tilde{H}_{\tau p}(s,z,w)\,\varphi(z)\big) = \lim_{\epsilon\to 0}e^{-\epsilon\Box_{\tau p}}\big[M^{z,\cdot}_{\tau p}\big(Y^J\tilde{H}_{\tau p}(s,\cdot,w)\,\varphi(\cdot)\big)\big](z).$$

Let $\tilde{\nabla} = \big(Z_{\tau p}+\overline{Z}_{\tau p}, i(Z_{\tau p}-\overline{Z}_{\tau p})\big)$. By Theorem 4.10 and the inductive hypothesis, we have

$$\begin{aligned}
|M^{z,w}_{\tau p}Y^J\tilde{H}_{\tau p}(s,z,w)| &\lesssim \lim_{\epsilon\to 0}\frac{\Lambda(z,\Delta)}{\delta}\Big(\|\tilde{H}_{\tau p}(s,\cdot,w)\varphi\|_{L^2} + \delta^2\|\Box_{\tau p}\big(\tilde{H}_{\tau p}(s,\cdot,w)\big)\varphi\|_{L^2} + \delta^2\|\tilde{H}_{\tau p}(s,\cdot,w)|\nabla^2\varphi|\|_{L^2}\Big)\\
&\lesssim \lim_{\epsilon\to 0}\Lambda(z,s^{1/2})\big(|Y^J\tilde{H}_{\tau p}(s,z,w)| + \delta|\tilde{\nabla}Y^J\tilde{H}_{\tau p}(s,z,w)| + \delta^2|\Box_{\tau p,z}Y^J\tilde{H}_{\tau p}(s,z,w)|\big)\\
&\lesssim \Lambda(z,s^{1/2})e^{-c\frac{|z-w|^2}{s}}\Lambda(z,s^{1/2})^{n-1}e^{-c\frac{|z-w|}{\mu_p(w,1/\tau)}}e^{-c\frac{|z-w|}{\mu_p(z,1/\tau)}}\max\left\{\frac{1}{s^{1+\frac{\ell}{2}}}, \frac{1}{\mu_p(w,1/\tau)^{2+\ell}}\right\}.
\end{aligned}$$

By the argument leading to Remark 7.8, it is enough to only check (n,ℓ)-derivatives of the form $M_{\tau p}Y^J$. $\square$

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